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Coarse embeddings of graphs into Hilbert space

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Abstract. In this paper, we study coarse embeddings of graphs into Hilbert space. For graph Γ expressible as an infinite union of coarsely embeddable subgraphs, Γ_i, we prove that if the nerve of the covering of Γ by the Γ_i is a tree and any nonempty intersections of the subgraphs have universally bounded diameter then Γ is coarsely embeddable into a Hilbert space.

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1 Introduction

The notion of uniform or coarse embeddability of metric spaces into Hilbert space was introduced by Gromov [3] and by Ferry, Ranicki, and Rosenberg [2] in connection with the study of the Novikov conjecture for discrete groups. One reason for the recent interest in coarse embeddings into Hilbert space is the result by Yu [9], which implies that a discrete metric space with bounded geometry that coarsely embeds into Hilbert space satisfies the coarse Baum-Connes conjecture. Applying this result to a finitely generated group, G, whose classifying space has the homotopy type of a finite CW complex and whose Cayley graph with respect to any finite generating set coarsely embeds into Hilbert space implies that the strong Novikov conjecture holds for G.

Recently, more work has been done on obstructions to the coarse embedding of graphs and general metric spaces into Hilbert space. Ostrovskii [4] and Tessera [8] characterize non-embeddability into Hilbert space in terms of a family of subgraphs exhibiting expander-like properties, and Ostrovskii [5] further shows that graphs with no K_r minors coarsely embed into Hilbert space. In another direction, Ren [6] and Dadarlat and Guentner [1] study metric spaces expressible as the union of coarsely embeddable subspaces and establish conditions on the subspaces that ensure the embeddabilty of the overall metric space.

In this paper, we continue along the lines of Ren, Dadarlat and Guentner by establishing sufficient conditions for graphs to be coarsely embeddable into a Hilbert space by investigating infinite coverings by subgraphs for which the nerve of the cover is a tree and the diameters of intersections of subgraphs are universally bounded. The size restriction on the intersections of subgraphs eliminates the necessity that the intrinsic metric on the subgraphs agree with the metric inherited from the full graph. Specifically, we show the following.

Main Theorem. *Let* $\Omega = \{\Gamma_1, \Gamma_2, \ldots\}$ *be a covering of a connected graph* Γ *by subgraphs and let* $\mathcal{N}(\Omega)$ *be the 1-skeleton of the nerve of the cover. If*

1. $\mathcal{N}(\Omega)$ *is a tree,*
2. *there exists* $N \in \mathbb{N}$ *such that for all* i, j, $\Gamma_i \cap \Gamma_j$ *is empty or of diameter less than* N,
3. *there exist coarse embeddings* $f_i : \Gamma_i \to \mathcal{H}$ *into a real Hilbert space* $\mathcal{H}$, *universally controlled by* $\rho_1 < \rho_2$,

then Γ *is coarsely embeddable into the Hilbert space* $\mathcal{H} \oplus \ell^2(\mathbb{N})$.

In Section 2 we review background on coarse embeddings and Hilbert spaces. In Section 3, we start by showing that a connected graph is coarsely-embeddable into Hilbert space if it is covered by infinitely many coarsely-embeddable subgraphs when we impose three conditions: the subgraphs intersect pairwise in at most one vertex, the coarse-embeddings for each subgraph are uniformly controlled, and the nerve of the cover is a tree. We then generalize this with Lemma 1 in order to obtain the Main Theorem.

2 Background and notation

We first recall some definitions from coarse geometry and refer the reader to the lectures of Roe [7] for a complete introduction to the subject. Throughout this paper, we view each graph as a metric space with metric induced by giving each edge length one. This is commonly referred to as the "edge metric".

Definition 1. *Let X and Y be metric spaces with metrics d_X and d_Y, respectively. A function $f : X \to Y$ is a* coarse embedding *if there exist non-decreasing functions $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \rho_1(t) = \infty$ and, for all $a, b \in X$,*

$$\rho_1(d_X(a,b)) \leq d_Y(f(a), f(b)) \leq \rho_2(d_X(a,b)).$$

We refer to ρ_1 and ρ_2 as the lower and upper controlling functions *respectively.*

Intuitively, a coarse embedding is a map between metric spaces where the change in distance is controlled by non-decreasing functions. While this is a weaker condition than quasi-isometry, coarse-embeddability from one metric space to another signals that distance is "coarsely" similar in these spaces.

For an example of a coarse-embedding that is not a quasi-isometry, let Y be the Cayley graph of the Baumslag-Solitar group $\mathrm{BS}(1,2) = \langle a, b \mid bab^{-1} = a^2\rangle$ with the edge metric and let X be the subgraph of Y corresponding to the cyclic subgroup $\langle a\rangle$, also with the edge metric. If $f : X \to Y$ is the inclusion map, then for all $a, b \in X$ the distance in the image does not increase, so that $d_Y(f(a), f(b)) \leq d_X(a,b)$ and we can set $\rho_2 = \mathrm{id}_{\mathbb{R}}$ the identity. However, distance in the image of f is not controlled below by the identity because of the relation in the group $bab^{-1} = a^2$. It turns out that $p_1(t) = \log_2(t)$ is a suitable lower controlling function.

Many of our arguments involve selecting a geodesic between a pair of vertices x and y. Though there may be more than one such geodesic, we generally fix one and use the notation $[x,y]$ to denote this fixed geodesic. If z is a vertex on the geodesic $[x,y]$, then $[x,y]$ consists of two geodesics of the form $[x,z]$ and $[z,y]$. We will use the notation $[x,y] = [x,z] \cup [z,y]$ in this situation. When we have need to use the vertices along the geodesic $[x,y]$, we write $x = v_0, v_1, v_1, \ldots, v_{n-1}, v_n = y$ with $d(v_i, v_{i+1}) = 1$.

A Hilbert space $\mathcal{H}$ is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. We will use the notation $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}\oplus\ell^2(\mathbb{N})}$ to denote the norms in the Hilbert spaces $\mathcal{H}$, resp., $\mathcal{H}\oplus\ell^2(\mathbb{N})$, where

$$\ell^2(\mathbb{N}) := \left\{ a : \mathbb{N} \to \mathbb{R} \mid \sum_{n=1}^{\infty} |a(n)|^2 < \infty \right\}$$

is the real Hilbert space of square-summable sequences endowed with inner product $\langle a, b\rangle = \sum_{n=1}^{\infty} a(n)b(n)$. Note that coarse embeddability into a real Hilbert space is equivalent to coarse embeddability into a complex one. Therefore, throughout this paper, unless otherwise noted, all Hilbert spaces are real.

Finally, we use $V(\Gamma)$ to refer to the vertex set of a graph Γ and $E(\Gamma)$ to refer to the edge set of Γ. Since $V(\Gamma)$ is coarsely equivalent (indeed, quasi-isometric) to Γ as a metric space, we may focus on embedding only $V(\Gamma)$. Therefore, unless otherwise stated all coarse embeddings of graphs will have as domain the vertex set of the graph in question. In particular, we slightly abuse notation and use $f : \Gamma \to \mathcal{H}$ to refer to a function from $V(\Gamma)$ to $\mathcal{H}$.

3 Infinite Covers of Graphs

Let $\Omega = \{\Gamma_1, \Gamma_2, \ldots\}$ be a cover of a connected graph Γ by subgraphs Γ_i. To each cover of this form Ω, we may define a graph $\mathcal{N}(\Omega)$ by taking vertex set in one-to-one correspondence with elements in Ω. Two vertices have an edge between them if their corresponding subgraphs intersect. This is called the "1-skeleton" of a more complicated construction called the "nerve of the cover" that results in a simplicial complex. Our first theorem in this section deals with the primary difficulties of constructing a coarse embedding of an infinite union of subgraphs from coarse embeddings of the subgraphs themselves.

Theorem 1. *Let $\mathcal{H}$ be a real Hilbert space and let $\Omega = \{\Gamma_1, \Gamma_2, \ldots\}$ be a covering of a connected graph Γ by subgraphs such that $|\Gamma_i \cap \Gamma_j| \leq 1$ for all $\Gamma_i, \Gamma_j \in \Omega$. If there exist coarse embeddings $f_i : \Gamma_i \to \mathcal{H}$ uniformly controlled by $\rho_1 < \rho_2$ and if $\mathcal{N}(\Omega)$ is a tree, then Γ is coarsely embeddable into the Hilbert space $\mathcal{H} \oplus \ell^2(\mathbb{N})$.*

In order to prove Theorem 1 we fix a vertex x_0 as base point, and consider distance inside subgraphs along a geodesic to the base point. Specifically, for $x \in V(\Gamma)$ denote by $\delta_{i,x}$ the distance a geodesic from x to x_0 travels through the subgraph Γ_i in Ω. This quantity is well-defined since $\mathcal{N}(\Omega)$ is a tree and intersections of subgraphs are at most one vertex. There may be more than one geodesic between x and x_0 but the length of all all of them within a particular subgraph Γ_i in Ω are the same. Again, since intersections of subgraphs in Ω are at most single vertices, for any $x \in V(\Gamma)$, all but finitely many $\delta_{i,x}$ are equal to 0. This holds true even if either or neither Γ nor $\mathcal{N}(\Omega)$ is locally finite.

For vertices x and y, fix geodesics $[x, x_0]$ and $[y, x_0]$. We refer to first subgraph along $[x, x_0]$ (starting at x) that contains a point of $[y, x_0]$ as the "branching space" of x and y. Note that we would get the same space if we used the geodesic $[y, x_0]$ instead. Moreover, the branching space does not depend on the particular geodesics selected, and since $\mathcal{N}(\Omega)$ is a tree and intersections of subgraphs contain at most one vertex, the only subgraph that contains both a portion of a geodesic from x to x_0 and a portion of a geodesic from y to x_0 such that $\delta_{i,x}$ differs from $\delta_{i,y}$ is the branching space of x and y (though they may, of course, be equal in some cases).

The proof requires that the coarse maps agree on intersections of subgraphs and we begin by showing that having singleton intersections allows us to assume this without loss of generality.

Proof of Theorem 1. We first remark that since $\mathcal{N}(\Omega)$ is a tree and all nonempty intersections of subgraphs Γ_i in Ω are singletons, the subgraphs Γ_i are actually isometrically embedded. That is, $d_{\Gamma_i} = d_\Gamma|_{\Gamma_i}$. Therefore, we use d to denote both distance in Γ and distance in any Γ_i.

We begin the proof by translating the images of the f_i so that our coarse maps agree on intersections of subgraphs as follows. Denote the vertex set of $\mathcal{N}(\Omega)$ by $\{a_1, a_2, \ldots\}$, with a_i corresponding to Γ_i. Fix Γ_1 as a base graph, and for $\ell \in \mathbb{N}\cup\{0\}$ set $S_\ell = \{\Gamma_i \mid d_{\mathcal{N}}(a_i, a_1) \leq \ell\}$, where $d_{\mathcal{N}}$ denotes distance in the nerve. We define new coarse maps recursively by first setting $f_1' = f_1$ and then assuming that the f_i' have been defined for all $\Gamma_i \in S_M$ for some $M \in \mathbb{N} \cup \{0\}$. Because $\mathcal{N}(\Omega)$ is a tree, for any $\Gamma_j \in S_{M+1}$ there exists a unique Γ_k such that $\Gamma_k \in S_M$ and $\Gamma_j \cap \Gamma_k$ is nonempty. Define f_j' by $f_j'(x) = f_j(x) - f_j(z_j) + f_k'(z_j)$ if $\Gamma_j \cap \Gamma_k = \{z_j\}$. Continuing this way yields coarse embeddings $f_1', f_2', \ldots$ that agree on any nonempty intersections and are clearly controlled by the same controlling functions as the original embeddings f_i. To keep the notation simple, we replace each f_i by f_i', so we have $f_i(x) = f_j(x)$ if $x \in \Gamma_i \cap \Gamma_j$.

We remark that local finiteness of $\mathcal{N}(\Omega)$ is not required for this argument. Even if $S_{m+1} \setminus S_m$ is infinite all f_k' for $a_k \in S_{m+1} \setminus S_m$ may be defined independently and simultaneously at step $m+1$.

Next define $g : \Gamma \to \mathcal{H} \oplus l^2(\mathbb{N})$ by

$$g(x) = (f_i(x), \delta_{1,x}, \delta_{2,x}, \ldots)$$

for $x \in \Gamma_i$, which is well defined since $f_i(x) = f_j(x)$ whenever $x \in \Gamma_i \cap \Gamma_j$ and only finitely many $\delta_{i,x}$ are nonzero. We construct functions θ_1, θ_2 that we prove below to be controlling functions for g. As in the case of a finite cover, we may replace a lower controlling function with a smaller nondecreasing function (with limit ∞) and an upper controlling function with a larger nondecreasing function. Thus, without loss of generality, we may assume that $\rho_1(t) \leq t \leq \rho_2(t)$. For ease of notation, set $\rho_2(1) = m$. Define $\theta_1' : [0, \infty) \to [0, \infty)$ by

$$\theta_1'(t) = \frac{\rho_1\left(\frac{2mt}{2m+1}\right)}{4m}$$

Set $\theta_1(t) = \min\{\theta_1'(t), \sqrt{\theta_1'(t)}\}$. Since ρ_1 is non-decreasing and $\lim_{t\to\infty} \rho_1(t) = \infty$ it is clear that θ_1 is non-decreasing and $\lim_{t\to\infty} \theta_1(t) = \infty$. Now define $\theta_2 : [0, \infty) \to [0, \infty)$ by $\theta_2(t) = \max\{(tm + t, \rho_2(t) + t\}$. It is clear that θ_2 is non-decreasing. It remains to show that for all $x, y \in \Gamma$,

$$\theta_1(d(x, y)) \leq ||g(x) - g(y))||_{\mathcal{H}\oplus l^2(\mathbb{N})} \leq \theta_2(d(x, y)). \quad (1)$$

There are two cases: $x, y \in \Gamma_i$ for some i and $x \in \Gamma_k \setminus \Gamma_j$, $y \in \Gamma_j \setminus \Gamma_k$ for some j and k with $j \neq k$.

For the first case, $x, y \in \Gamma_i$ for some i, we begin by establishing the left side of inequality

(1). We have,

$$\begin{aligned}\|g(x)-g(y)\|_{\mathcal{H}\oplus l^2(\mathbb{N})} &= \sqrt{\|f_i(x)-f_i(y)\|^2_{\mathcal{H}} + \sum_{i=1}^{\infty}(\delta_{i,x}-\delta_{i,y})^2} \\ &\geq \|f_i(x)-f_i(y)\|_{\mathcal{H}} \\ &\geq \rho_1(d(x,y)) \geq \theta_1'(d(x,y)) \geq \theta_1(d(x,y)),\end{aligned}$$

since ρ_1 is non-decreasing and $\frac{2mt}{2m+1} < t$.

For the right half of Inequality (1), observe that since $\mathcal{N}(\Omega)$ is a tree and $|\Gamma_i \cap \Gamma_j| \leq 1$, we have that $\sum_{i=1}^{\infty} |\delta_{i,x} - \delta_{i,y}| \leq d(x,y)$. This is because of four facts. First, $\delta_{i,x} = \delta_{i,y} = 0$ for any Γ_i not containing a portion of either geodesic $[x_0, x]$ or $[x_0, y]$. Second, for Γ_i along $[x_0, x]$ before the branching space (and therefore along $[x_0, y]$ before the branching space), $\delta_{i,x} = \delta_{i,y}$. Third, for subgraphs along $[x_0, x]$ after the branching space, $\delta_{i,y} = 0$ and $\delta_{i,x}$ is the length of $[x, y]$ within Γ_i, and similarly for Γ_i along $[x_0, y]$ after the branching space. Finally, by the triangle inequality, if the branching space is Γ_i, then $|\delta_{i,x} - \delta_{i,y}|$ is less than or equal to the length of the portion of the geodesic $[x, y]$ that lies in the branch space. Thus,

$$\begin{aligned}\|g(x)-g(y)\|_{\mathcal{H}\oplus l^2(\mathbb{N})} &= \sqrt{\|f_i(x)-f_i(y)\|^2_{\mathcal{H}} + \sum_{i=1}^{\infty}(\delta_{i,x}-\delta_{i,y})^2} \\ &\leq \|f_i(x)-f_i(y)\|_{\mathcal{H}} + \sum_{i=1}^{\infty}|\delta_{i,x}-\delta_{i,y}| \\ &\leq \rho_2(d(x,y)) + d(x,y) \leq \theta_2(d(x,y)),\end{aligned}$$

as required.

For the second case, suppose that $x \in \Gamma_k \setminus \Gamma_j$ and $y \in \Gamma_j \setminus \Gamma_k$. Fix geodesics $[x, x_0]$, $[y, x_0]$ and let α and β be the first points on our fixed geodesics $[x, x_0]$ and $[y, x_0]$ respectively, that belong to the branching space of x and y. These points do not depend on the specific geodesics selected because intersections are singletons and there is only one branching space for each pair of vertices x and y. Fix a geodesic $[\alpha, \beta]$ and let $[x, y]$ refer to the geodesic from x to y given by $[x, \alpha] \cup [\alpha, \beta] \cup [\beta, y]$.

For the right inequality of inequality (1), let

$$x = z_0, z_1, \ldots, z_s, \ldots, z_n, \ldots, z_{r-1}, z_r = y$$

be the list of vertices along $[x, y]$ with $\alpha = z_s$ and $\beta = z_n$. For $1 \leq i \leq r$, we may select subgraphs $\Delta_i = \Gamma_{\kappa_i}$ such that $z_i \in \Delta_i \cap \Delta_{i-1}$. Note that Δ_s is the branching space of x and y and for $s \leq i \leq n-1$ we may select $\Delta_i = \Delta_s$. We denote the coarse map $f_{\kappa_i} : \Delta_i \to \mathcal{H}$ by

h_i and we use that $\sum\limits_{i=1}^{\infty} |\delta_{i,x} - \delta_{i,y}| \leq d(x,y)$. Now,

$$\begin{aligned}
\|g(x) - g(y)\|_{\mathcal{H}\oplus\ell^2(\mathbb{N})} &= \sqrt{\|f_k(x) - f_j(y)\|_{\mathcal{H}}^2 + \sum_{i=1}^{\infty}(\delta_{i,x} - \delta_{i,y})^2} \\
&\leq \|f_k(x) - f_j(y)\|_{\mathcal{H}} + \sum_{i=1}^{\infty} |\delta_{i,x} - \delta_{i,y}| \\
&\leq \sum_{i=1}^{r} \|h_{i-1}(z_{i-1}) - h_i(z_i)\|_{\mathcal{H}} + d(x,y) \\
&\leq \sum_{i=1}^{r} \rho_2(d(z_{i-1}, z_i)) + d(x,y) \\
&\leq r\rho_2(1) + d(x,y) \\
&= d(x,y)m + d(x,y) \leq \theta_2(d(x,y)),
\end{aligned}$$

as required.

For the left inequality of inequality (1), suppose that we have $d(x,\alpha) > \theta_1'(d(x,y))$. By the definition of the branching space, the geodesic $[y,x_0]$ meets none of the subgraphs through which the geodesic $[x,\alpha]$ passes. Hence, $\sum_{i=1}^{\infty} |\delta_{i,x} - \delta_{i,y}| > d(x,\alpha)$. This means that $\sum_{i=1}^{\infty} |\delta_{i,x} - \delta_{i,y}| \geq \theta_1'(d(x,y))$. Finally, since Γ is a graph, and we focus only on the vertex set, all distances are integers. So squaring a distance will not decrease it. Thus,

$$\begin{aligned}
\|g(x) - g(y)\|_{\mathcal{H}\oplus\ell^2(\mathbb{N})} &= \sqrt{\|f_1(x) - f_1(y)\|_{\mathcal{H}}^2 + \sum_{i=1}^{\infty}(\delta_{i,x} - \delta_{i,y})^2} \\
&\geq \sqrt{\sum_{i=1}^{\infty}(\delta_{i,x} - \delta_{i,y})^2} \geq \sqrt{\sum_{i=1}^{\infty} |\delta_{i,x} - \delta_{i,y}|} \\
&\geq \sqrt{d(x,\alpha)} > \sqrt{\theta_1'(d(x,y))} \geq \theta_1(d(x,y)),
\end{aligned}$$

as required. An equivalent argument works in the case $d(y,\beta) > \theta_1'(d(x,y))$.

Next suppose that $d(x,\alpha) \le \theta_1'(d(x,y))$ and $d(y,\beta) \le \theta_1'(d(x,y))$. We have,

$$\begin{aligned}
\|g(x)-g(y)\|_{\mathcal{H}\oplus\ell^2(\mathbb{N})} &= \sqrt{\|f_k(x)-f_j(y)\|_{\mathcal{H}}^2 + \sum_{i=1}^{\infty}(\delta_{i,x}-\delta_{i,y})^2} \\
&\ge \|f_k(x)-f_j(y))\|_{\mathcal{H}} \\
&\ge \|f_s(\alpha)-f_s(\beta)\|_{\mathcal{H}} - \|f_s(\alpha)-f_k(x)\|_{\mathcal{H}} - \|f_j(y)-f_s(\beta)\|_{\mathcal{H}} \\
&\ge \rho_1(d(\alpha,\beta)) - \sum_{i=1}^{s}\|h_{i-1}(z_{i-1})-h_i(z_i)\|_{\mathcal{H}} \\
&\quad - \sum_{i=n+1}^{r}\|h_{i-1}(z_{i-1})-h_i(z_i)\|_{\mathcal{H}} \\
&\ge \rho_1(d(x,y)-d(x,\alpha)-d(y,\beta)) - \rho_2(1)(d(x,\alpha)+d(y,\beta)) \\
&\ge \rho_1(d(x,y)-2\theta_1'(d(x,y))) - 2m\theta_1'(d(x,y)),
\end{aligned}$$

where the last two inequalities follow from the fact that ρ_1 is increasing, and the way the h_i's are constructed. Indeed, $h_i(z_i) = h_{i-1}(z_i)$ for all $i \ge 1$, so that the distance between any pair $||h_i(z_i) - h_{i-1}(z_{i-1})||_{\mathcal{H}}$ is at most $\rho_2(1)$. We claim that the final expression is greater than $\theta_1(d(x,y))$. To see this, observe that:

$$\begin{aligned}
\rho_1(t-2\theta_1'(t)) - 2m\theta_1'(t) &= \rho_1\left(t - 2\frac{\rho_1\left(\frac{2mt}{2m+1}\right)}{4m}\right) - 2m\frac{\rho_1\left(\frac{2mt}{2m+1}\right)}{4m} \\
&\ge \rho_1\left(t - \frac{\left(\frac{2mt}{2m+1}\right)}{2m}\right) - \frac{1}{2}\rho_1\left(\frac{2mt}{2m+1}\right) \\
&= \frac{1}{2}\rho_1\left(\frac{2mt}{2m+1}\right) \\
&\ge \frac{1}{2m}\rho_1\left(\frac{2mt}{2m+1}\right) = \theta_1'(t) \ge \theta_1(t),
\end{aligned}$$

where the first inequality uses the facts that $\rho_1(t) \le t$ and ρ_1 is nondecreasing.

This finishes the proof of inequality (1) and the proof that Γ is coarsely embeddable into $\mathcal{H} \oplus \ell^2(\mathbb{N})$. $\square$

We next extend Theorem 1 to a covering with intersections larger than a single vertex by quotienting out by the sets of intersections. Recall that if Γ is a graph and $\Psi = \{S_1, S_2, \ldots\}$ is a partition of $V(\Gamma)$, the quotient of Γ by Ψ is the graph Γ/Ψ with vertex set $V(\Gamma/\Psi) = \Psi$ and edge set containing an edge (S_i, S_j) whenever Γ contains an edge (x,y) with $x \in S_i$ and $y \in S_j$.

Lemma 1. *For graphs Γ and Δ, let $f : V(\Gamma) \to V(\Delta)$ be a function with the property that if a vertex x is adjacent to a vertex y in Γ, then $f(x)$ is adjacent to $f(y)$ in Δ or $f(x) = f(y)$.*

If there exists $M \in \mathbb{N}$ such that for each edge $e = (x, y)$ of Δ, the diameter of $f^{-1}(\{x, y\})$ is less than M, then f is a coarse embedding.

Proof. Let $x, y \in V(\Gamma)$. We consider two cases: $f(x) = f(y)$ and $f(x) \neq f(y)$. We claim that if $f(x) = f(y)$, then $d_\Gamma(x, y) \leq M$. Indeed, if $z \in V(\Delta)$ is adjacent to $f(x)$ (and $f(y)$), then by hypothesis the diameter of $f^{-1}(\{z, f(x)\}) = f^{-1}(\{z, f(y)\})$ is less than M. Conclude that $d_\Gamma(x, y) \leq M$, as needed.

If $f(x) \neq f(y)$, fix a geodesic $[f(x), f(y)]$ in Δ. By considering an edge path in Γ constructed by concatenating, in order, geodesics between preimages of the vertices on $[f(x), f(y)]$, the triangle inequality yields $d_\Gamma(x, y) \leq M \cdot d_\Delta(f(x), f(y))$. Conclude that for any $x, y \in V(\Gamma)$, we have that

$$d_\Gamma(x, y) \leq \max\{M \cdot d_\Delta(f(x), f(y)),\ M\} \leq M \cdot d_\Delta(f(x), f(y)) + M,$$

so that $\max\{0,\ d(x, y)/M - 1\} \leq d_\Delta(f(x), f(y))$. Since f preserves adjacency, it cannot increase distance, so

$$\max\{0,\ d(x, y)/M - 1\} \leq d_\Delta(f(x), f(y)) \leq d(x, y),$$

as needed to show that f is a coarse embedding. □

We now have,

Main Theorem. *Let $\Omega = \{\Gamma_1, \Gamma_2, \ldots\}$ be a covering of a connected graph Γ by subgraphs such that*

1. *$\mathcal{N}(\Omega)$ is a tree.*

2. *There exists $N \in \mathbb{N}$ such that for all i, j, $\Gamma_i \cap \Gamma_j$ is empty or of diameter less than N.*

3. *There exist coarse embeddings $f_i : \Gamma_i \to \mathcal{H}$ universally controlled by $\rho_1 < \rho_2$.*

Then Γ is coarsely embeddable into the Hilbert space $\mathcal{H} \oplus \ell^2(\mathbb{N})$.

Proof. Since $\mathcal{N}(\Omega)$ is a tree, if i, j and k are distinct then $\Gamma_i \cap \Gamma_j \cap \Gamma_k = \emptyset$. Thus, each vertex v of Γ belongs to at most two subgraphs Γ_i. Let Ψ be the partition of $V(\Gamma)$ consisting of singleton sets for each v in $V(\Gamma)$ that is contained in only one Γ_i together with all nonempty sets of the form $\Gamma_i \cap \Gamma_j$. The quotient map $q : \Gamma \to \Gamma/\Psi$ satisfies the conditions of Lemma 1 with $M = 2N + 1$.

Since $\mathcal{N}(\Omega)$ is a tree, setting $\Gamma'_i = q(\Gamma_i)$ gives a covering $\Omega' = \{\Gamma'_1, \Gamma'_2 \ldots\}$ of Γ/Ψ with the same nerve as Ω. Thus $\mathcal{N}(\Omega')$ is a tree and for each i, j, $|\Gamma'_i \cap \Gamma'_j| \leq 1$. By Theorem 1, Γ/Ψ therefore coarsely embeds into $\mathcal{H} \oplus \ell^2(\mathbb{N})$. Composing a coarse embedding of Γ/Ψ with q gives a coarse embedding of Γ into $\mathcal{H} \oplus \ell^2(\mathbb{N})$ □

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