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Author's note on *Calculation of the Killing Form of a Simple Lie Group*

The following article was written as an unpublished note as a companion to the authors 1987 paper *Volumes of subgroups of compact Lie groups*. The note was included in the Rose MSTR series to make it available on the internet.

S. Allen Broughton 5 Aug 14

CALCULATION OF THE KILLING FORM OF SIMPLE LIE ALGEBRA

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Let L be a simple, complex Lie algebra, H a Cartan subalgebra $\Delta \subseteq H^*$ the root system. Let κ be the restriction of the Killing form of L to H and κ^* dualized form on H^* . These forms satisfy:

$$\kappa^*(x^*, y^*) = x^*(y) = \kappa(x, y), \quad x, y \in H,$$

where $x^* \in H^*$ is the element defined by

$$x^*(y) = \kappa(x, y) \quad y \in H.$$

The invariant form on H^* (which is unique up to multiplication by a scalar) is frequently normalized by requiring short roots to have length 1. If (\cdot, \cdot) denotes this normalized form then

$$\kappa^*(\lambda, \mu) = \epsilon(\lambda, \mu)$$

where $\epsilon = \kappa^*(\alpha, \alpha)$, α a short root. The Killing form is determined once ϵ is known.

A list of the values of ϵ is given in Table A, p. 527 of Freudenthal & de Vries monograph [F-V].

Before calculating ϵ below we need to establish some notation. For $\lambda, \mu \in H^*$ let $\langle \lambda, \mu \rangle = 2\kappa^*(\lambda, \mu) / \kappa^*(\mu, \mu) = 2(\lambda, \mu) / (\mu, \mu)$. Let $\alpha_1, \dots, \alpha_r$ be a basis of simple roots of Δ . There is a unique long root α^ℓ and a unique short root α^s such that $\langle \alpha^\ell, \alpha_i \rangle \leq 0, \langle \alpha^s, \alpha_i \rangle \leq 0$ ($\alpha^s = \alpha^\ell$ if there is only one root length). Let $\alpha_0 = \alpha^\ell$ then there are unique integers $m_0^\ell, \dots, m_r^\ell$ with $m_0^\ell = 1$ such that $\sum_{i=0}^r m_i^\ell \alpha_i = 0$. Let m_0^s, \dots, m_r^s be similarly defined. That $h^\ell = \sum_{i=0}^r m_i^\ell$ equals the Coxeter number is well-known.

By analogy let us call $h^S = \sum_{i=0}^r m_i^S$ the short Coxeter number. The number of roots in Δ is given by

$$(1) \quad |\Delta| = r \cdot h^S$$

Set $\alpha_0 = \alpha^l$ and construct an extended Dynkin diagram as follows.

Adjoin a node, α_0 , to the ordinary diagram. For each α_i , $i > 1$ join α_0 to α_i by a bond of strength $\langle \alpha_0, \alpha_i \rangle \langle \alpha_i, \alpha_0 \rangle$, draw an arrow from a longer to a shorter root, and write m_i^l above the node α_i . For the root α^S we get a similar diagram. The extended diagrams are recorded in Table 1 where the node α_0 is blackened. They can be constructed using Table 2, p. 66 of [H]. We denote the extended diagrams corresponding to long (short) roots of $A_r, B_r, C_r, D_r, E_r, F_4, G_2$ by $\tilde{A}_r, \tilde{B}_r^l, \tilde{C}_r^l, \tilde{D}_r, \tilde{E}_r, \tilde{F}_4^l, \tilde{G}_2^l$ ($\tilde{A}_r, \tilde{B}_r^S, \tilde{C}_r^S, \tilde{D}_r, \tilde{E}_r, \tilde{F}_4^S, \tilde{G}_2^S$).

Proposition. Let L be a simple Lie algebra over \mathbb{C} of rank r , H a CSA, $\Delta \subseteq H^*$ the root system and κ^* the dualized Killing form on H^* . Let $\alpha^S, \alpha^l \in \Delta$ be as above and let $h^S(\Delta^V)$ be the short Coxeter number of the dual root system. Then

$$\kappa^*(\alpha^S, \alpha^S) = ((\alpha^l, \alpha^l) h_S(\Delta^V))^{-1}$$

In Table 2 we give the values of $h^S(\Delta^V)$, (α^l, α^l) and ϵ^{-1} for each algebra.

Proof. Using the notation of [H], let, for $\lambda \in H^*$, $t_\lambda \in H$ be the element such that $\lambda(h) = \kappa(t_\lambda, h)$, $h \in H$, i.e. $\lambda = (t_\lambda)^*$. Let $h_\alpha = 2t_\alpha / \kappa(t_\alpha, t_\alpha)$ then $\kappa^*(\alpha, \alpha) = \kappa(t_\alpha, t_\alpha) = 4/\kappa(h_\alpha, h_\alpha)$, and $\beta(h_\alpha) = 2\kappa(t_\beta, t_\alpha) / \kappa(t_\alpha, t_\alpha) = 2\kappa^*(\beta, \alpha) / \kappa^*(\alpha, \alpha) = \langle \beta, \alpha \rangle$. If α is a long root then $(\alpha^l, \alpha^l)_\epsilon = (\alpha, \alpha)_\epsilon = 4/\kappa(h_\alpha, h_\alpha)$. Thus it suffices to show that, letting $h^S = h^S(\Delta^V)$,

$$(2) \quad \kappa(h_\alpha, h_\alpha) = 4h^S.$$

Table 1.

Extended Dynkin Diagrams


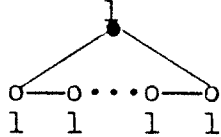
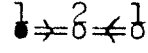
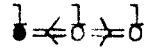
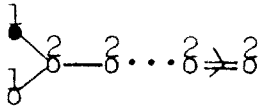
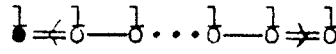
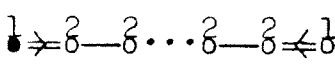
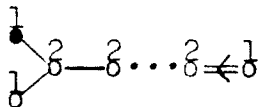
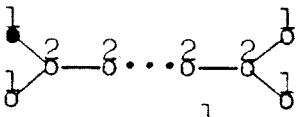
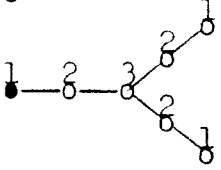
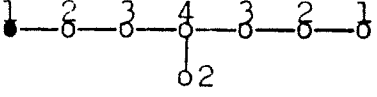
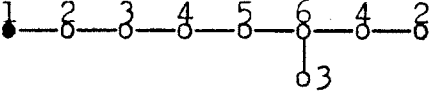
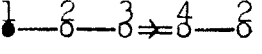
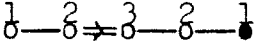
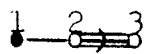
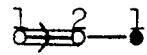
<u>Type</u>	<u>Long Root</u>	<u>Short Root</u>
A_1		same
$A_r \ r \geq 2$		same
B_2		
$B_r \ r \geq 3$		
$C_r \ r \geq 3$		
$D_r \ r \geq 4$		same
E_6		same
E_7		same
E_8		same
F_4		
G_2		

Table 2.

Algebra	$(\alpha^{\ell}, \alpha^{\ell})$	$h^S(\Delta^V)$	ϵ^{-1}
$A_r \quad \ell \geq 1$	1	$r + 1$	$r + 1$
$B_r \quad \ell \geq 2$	2	$2r - 1$	$4r - 2$
$C_r \quad \ell \geq 3$	2	$r + 1$	$2r + 2$
$D_r \quad \ell \geq 4$	1	$2r - 2$	$2r - 2$
E_6	1	12	12
E_7	1	18	18
E_8	1	30	30
F_4	2	9	18
G_2	3	4	12

Table 3.

Δ	$ \Delta $	R	$ R $
$A_r \quad r \geq 1$	$r(r + 1)$	A_{r-2}	$(r - 2)(r - 1)$
$B_r \quad r \geq 2$	$2r^2$	$A_1 \times B_{r-2}$	$2 + 2(r - 2)^2$
$C_r \quad r \geq 3$	$2r^2$	C_{r-1}	$2(r - 1)^2$
$D_r \quad r \geq 4$	$2r(r - 1)$	$A_1 \times D_{r-2}$	$2 + 2(r - 2)(r - 3)$
E_6	72	A_5	30
E_7	126	D_6	60
E_8	240	E_7	126
F_4	48	C_3	18
G_2	12	A_1	2

Since h_α acts diagonally with respect to a basis of root vectors with eigenvalues $\langle \beta, \alpha \rangle = \beta(h_\alpha)$, $\beta \in \Delta$, then $\kappa(h_\alpha, h_\alpha) = \sum_{\beta \in \Delta} \beta(h_\alpha)^2 = \sum_{\beta \in \Delta} \langle \beta, \alpha \rangle^2$. Since α is long then $\langle \beta, \alpha \rangle = \pm 1$ if $\beta \neq \pm \alpha$ and $\langle \beta, \alpha \rangle \neq 0$.

Let b be the cardinality of $\{\beta \mid \langle \beta, \alpha \rangle \neq 0\}$. As $\langle \alpha, \alpha \rangle^2 + \langle -\alpha, \alpha \rangle^2 = 8$ and there are $b - 2$ roots β with $\langle \beta, \alpha \rangle^2 = 1$, then

$$\kappa(h_\alpha, h_\alpha) = 6 + b.$$

Therefore, it suffices to show that

$$(3) \quad b = 4h^S - 6.$$

We need to compute the number of roots orthogonal to α . Since roots of the same length are conjugate under the Weyl group, the number $|\{\beta \in \Delta \mid (\alpha, \beta) = 0\}|$ depends only on the length of α .

Let $\alpha_0 = \alpha^\lambda$. Remove from the extended Dynkin diagram, corresponding to a long root, α_0 and all nodes connected to it, as well as bonds connected to removed nodes. This leaves us with a union of ordinary Dynkin diagrams. Let N denote the subset of $\{\alpha_1, \dots, \alpha_r\}$ so determined and denote by R the subroot system generated by N .

Claim: $R = \{\beta \in \Delta : (\alpha, \beta) = 0\}$.

Assuming the claim, we have $b = |\Delta| - |R|$. It then suffices by (3) to prove:

$$(4) \quad |\Delta| - |R| + 6 = 4h^S.$$

The proof of (4) follows immediately from Table 3 above. In turn, Table 3 is easily constructed from Table 1 and (1) by means of the claim.

Now to prove the claim: Let $\beta = \sum_{i=1}^r n_i \alpha_i$ be perpendicular to α_0 . Then $\sum_{i=0}^r n_i \langle \alpha_i, \alpha_0 \rangle = 0$. Since $\langle \alpha_i, \alpha_0 \rangle \leq 0$ and either all $n_i \geq 0$ or all $n_i \leq 0$, then all terms above are zero. Thus $n_i = 0$ if $\langle \alpha_i, \alpha_0 \rangle \neq 0$.

This implies that the support of β , $\text{supp}(\beta) = \{\alpha_i \mid n_i \neq 0\}$, lies in N .

Since the support of a root is always connected we may assume that $\text{supp}(\beta) \subseteq M$ the set of nodes of some connected component of the diagram determined by N . Assume β is negative, $\beta = \sum_{\alpha \in M} n_{\alpha} \alpha$, $n_{\alpha} \leq 0$ and let $\text{ht}(\beta) = \sum_{\alpha \in M} n_{\alpha}$. If $\langle \beta, \alpha_i \rangle > 0$ for some $\alpha_i \in M$ then $\sigma_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i$ is a negative root, perpendicular to α_0 , with support in M and such that $\text{ht}(\sigma_{\alpha_i}(\beta)) < \text{ht}(\beta)$. Iterating this process we arrive at γ with $\langle \gamma, \alpha_0 \rangle = 0$, $\langle \gamma, \alpha \rangle \leq 0$ $\alpha \in M$. It cannot happen that $\langle \gamma, \alpha \rangle = 0$ for all $\alpha \in M$.

Form an extended Dynkin diagram from γ and M . The resulting diagram is connected and must be one of the diagrams in Table 1. Let $\delta \in M \cup \{\gamma\}$ correspond to the blackened node, then $M \cup \{\gamma\} \setminus \{\delta\}$ is a basis for the root system generated by γ and M . Thus the diagram for M must be obtained by removing a non-disconnecting node, i.e. γ , from one of the diagrams in Table 1. Moreover the root γ , and hence β , belongs to the root system generated by M if and only if the removed node is a blackened node or one equivalent to it by an extended diagram symmetry. By inspection of Table 1, if γ is not in the root system generated by M , then the only possibilities for $M \cup \{\gamma\}$ and M are:

<u>$M \cup \{\gamma\}$</u>	<u>M</u>
\tilde{B}_r^{ℓ} $r \geq 3$	D_r
\tilde{C}_r^s	D_r
\tilde{E}_7	A_7
\tilde{E}_8	A_8, D_8
\tilde{F}_4^{ℓ}	B_4
\tilde{F}_4^s	C_4
\tilde{G}_2^{ℓ}	A_2
\tilde{G}_2^s	A_2

If there is a counterexample to the claim then one of the M 's listed above is a component of some N , and Δ will contain a subroot system generated by $M \cup \{\gamma\}$. If $M = A_2, A_7$ or A_8 then $\Delta = A_4, A_9$ or A_{10} by Table 3. (We are using the symbols A_r, B_r, \dots to denote a root system and the corresponding diagram.) It follows then that $G_2 \subseteq A_4, E_7 \subseteq A_9, E_8 \subseteq A_{10}$. The first is false since A_4 has only one root length while G_2 has 2. The last two fail because of order considerations. If $M = B_4$ then $F_4 \subseteq B_6$, an impossibility since the short roots of B_6 form an $A_1 \times A_1 \times A_1 \times A_1 \times A_1 \times A_1$ and the short roots of F_4 form a D_4 . For similar reasons we eliminate C_4 . If $M = D_4$ then we get $B_r \subseteq D_{r+2}, B_6 \subseteq E_7, C_r \subseteq D_{r+2}, C_6 \subseteq E_7$ or $E_8 \subseteq D_{10}$. The first four are rejected by root length considerations and the fifth by order considerations. All is now proven.

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- [F-V] Freudenthal, H. & de Vries, R., Linear Lie Groups, Academic Press, (1969).
- [H] Humphreys, J., Introduction to Lie Algebras and Representation Theory, Graduate Texts in Math #9, Springer-Verlag, (1970).