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Meghan Peer  
*Saginaw Valley State University*

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Meghan Peer <sup>a</sup>

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Rose-Hulman Institute of Technology

Department of Mathematics

Terre Haute, IN 47803

Email: [mathjournal@rose-hulman.edu](mailto:mathjournal@rose-hulman.edu)

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<sup>a</sup>Saginaw Valley State University

# TANGENT LINE AND TANGENT PLANE APPROXIMATIONS OF DEFINITE INTEGRALS

Meghan Peer

**Abstract.** Oftentimes, it becomes necessary to find approximate values for definite integrals, since the majority cannot be solved through direct computation. The methods of tangent line and tangent plane approximation can be derived as methods of integral approximation in two and three-dimensional spaces, respectively. Formulas are derived for both methods, and these formulas are compared with existing methods in terms of efficiency and error.

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# 1 Introduction to Definite Integrals and Approximation

A definite integral is defined as a limit of Riemann sums; therefore any Riemann sum could be used as an approximation to the integral. Consider a function  $f$  defined on a closed interval  $[a, b]$ . Divide the interval  $[a, b]$  into  $n$  subintervals of equal width  $\Delta x = \frac{b-a}{n}$ . We let  $x_0 (= a), x_1, x_2, \dots, x_n (= b)$  be the endpoints of these subintervals. The definite integral of  $f$  from  $a$  to  $b$  (illustrated by Figure 1 below) is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

provided that the limit exists, where  $x_i^*$  is any point in the  $i$ th subinterval  $[x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, n$  [3, p. 296]. If the limit does exist,  $f$  is said to be integrable on  $[a, b]$ .

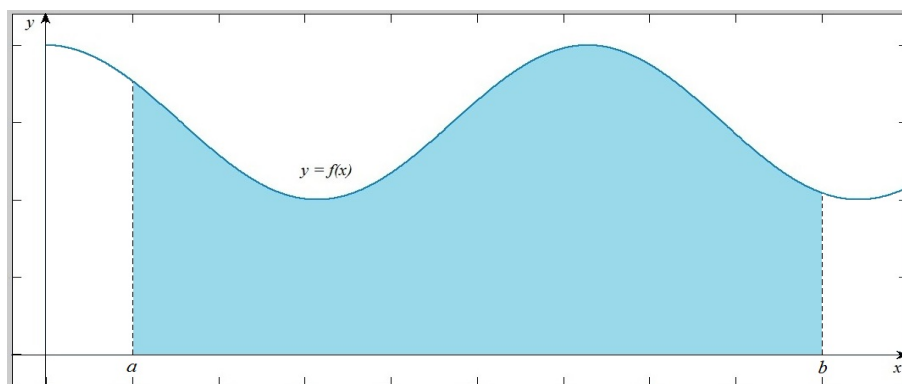


Figure 1: If  $f(x) \geq 0$ , the integral  $\int_a^b f(x)dx$  is the area under the curve  $y = f(x)$  from  $a$  to  $b$ .

The left endpoint approximation, right endpoint approximation, and Midpoint Rule all represent approximations of the definite integral by the use of rectangles. These three approximations are illustrated below for comparison purposes, where  $\Delta x = \frac{b-a}{n}$ , and  $\bar{x}_i$  is the midpoint of  $[x_{i-1}, x_i]$ . The approximations  $L_n$ ,  $R_n$ , and  $M_n$  are the left endpoint, right endpoint, and midpoint approximations, respectively:

$$\int_a^b f(x)dx \approx L_n = \sum_{i=1}^n f(x_{i-1})\Delta x$$

$$\int_a^b f(x)dx \approx R_n = \sum_{i=1}^n f(x_i)\Delta x$$

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^n f(\bar{x}_i)\Delta x. \quad (1.1)$$

The Trapezoidal Rule approximation,  $T_n$ , essentially is a result of averaging the right endpoint and left endpoint approximations. Figuratively, this method of approximation is derived from the accumulation of areas of trapezoids, rather than rectangles, and is given by

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]. \quad (1.2)$$

Another approximation technique, Simpson's Rule, approximates the definite integral using parabolas instead of straight line segments [3, p. 535]. This approximation,  $S_n$  is given by

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where  $n$  is an even number.

In this paper, we study an additional approximation technique. Specifically, we will approximate a definite integral of a function by using the tangent line approximation to the function at the left endpoint of the subintervals. We introduce this technique in Section 2.1, below, and compute a bound on its error in Section 2.2. In Section 2.3 we compare this error to the error in the Midpoint Rule and Trapezoidal Rule explicitly, and experimentally to the error in the left and right endpoint methods. We provide an explicit example in Section 2.4. In Section 3, we extend the ideas of Section 2 to consider approximations of definite integrals of functions of two variables. As such, we consider approximations of the definite integral by tangent planes, compute an error bound, and conclude with an example. Finally, we provide an introduction to a more expansive approach to the tangent line method, involving the approximation of a curve using any Taylor polynomial of degree  $n$ .

## 2 Tangent Line Approximation of Definite Integrals and Error Formula

### 2.1 Left Endpoint Tangent Line Approximation

In this section, we begin our study of using tangent lines in the calculation of definite integral approximations. Given a differentiable function  $f$ , we approximate  $f$  on each subinterval

$[x_i, x_{i+1}]$  by computing the tangent line to  $f$  at the left endpoint  $x_i$ . We then approximate the definite integral of  $f$  on the interval  $[a, b]$  by summing the integrals of these tangent line approximations. We will denote the tangent line on  $[x_i, x_{i+1}]$  by  $T_i(x)$ , so that

$$T_i(x) = f'(x_i)(x - x_i) + f(x_i)$$

and

$$\int_{x_i}^{x_{i+1}} T_i(x) dx = \int_{x_i}^{x_{i+1}} [f'(x_i)(x - x_i) + f(x_i)] dx.$$

Since we want the accumulated values of each integral for each subinterval, we must sum each integral of the form above. This gives

$$\begin{aligned} & \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} [f'(x_i)(x - x_i) + f(x_i)] dx \\ &= \sum_{i=0}^{n-1} \left[ f'(x_i) \left( \frac{(x_{i+1})^2}{2} - \frac{(x_i)^2}{2} - x_i(x_{i+1} - x_i) \right) + f(x_i)(x_{i+1} - x_i) \right] \\ &= \sum_{i=0}^{n-1} \left[ f'(x_i) \left( \frac{1}{2}(x_{i+1} + x_i)\Delta x - (x_i)\Delta x \right) + f(x_i)\Delta x \right] \\ &= \sum_{i=0}^{n-1} \left[ \Delta x \left( \frac{1}{2}f'(x_i)\Delta x + f(x_i) \right) \right]. \end{aligned}$$

Therefore, the formula for the tangent line approximation of definite integrals (illustrated by Figure 2 below), using left endpoints, is given by

$$\int_a^b f(x) dx \approx \sum_{i=0}^{n-1} \left[ \Delta x \left( \frac{1}{2}f'(x_i)\Delta x + f(x_i) \right) \right]. \quad (2.1)$$

Remark: The limitation to using this formula in calculations resides in the fact that in order to use (2.1) to approximate an integral, it is necessary that the given function is differentiable, since we must be able to compute the derivative.

Given this limitation, we can derive a similar formula by approximating the derivative, in which case, we have [3, p. 114]

$$f'(x_i) \approx \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}. \quad (2.2)$$

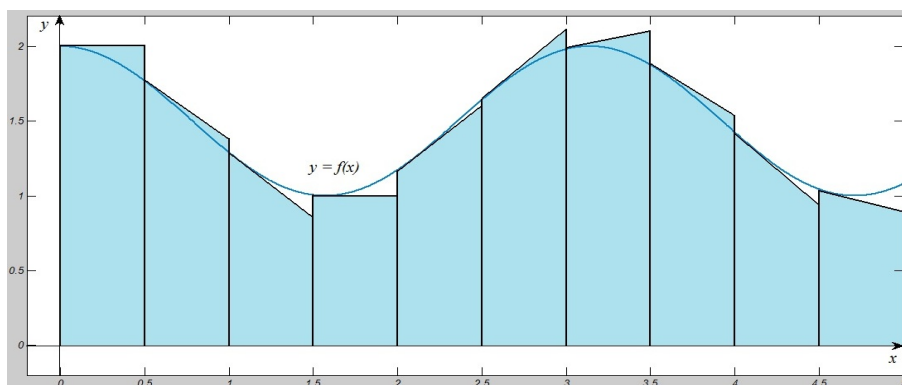


Figure 2: Tangent line approximation using left endpoints.

Substituting this into (2.1) gives

$$\begin{aligned}
 \int_a^b f(x)dx &\simeq \sum_{i=0}^{n-1} \left[ \Delta x \left( \frac{1}{2} \left[ \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right] \Delta x + f(x_i) \right) \right] \\
 &= \sum_{i=0}^{n-1} \left[ \Delta x \left( \frac{1}{2} (f(x_i) + f(x_i + \Delta x)) \right) \right] \\
 &= \sum_{i=0}^{n-1} \left[ \Delta x \left( \frac{1}{2} (f(x_i) + f(x_{i+1})) \right) \right].
 \end{aligned}$$

This formula is simply the average of the right endpoint approximation and the left endpoint approximation, which is the Trapezoidal Rule given in (1.2). Therefore, the tangent line approximation is equivalent to the Trapezoidal Rule when the derivative is approximated by (2.2).

## 2.2 Error Formula for Tangent Line Approximation

In this section, we compute a bound on the error in the tangent line approximation. Later, we will compare this error formula to the error bounds in the Trapezoidal and Midpoint rules. Note that Simpson's Rule provides the least error for smooth functions and is not a straight line approximation like the other methods described above. As such, we will not refer to Simpson's Rule in the following comparisons and examples.

The error bound for the tangent line approximation,  $E_T$ , is given in the following theorem.

**Theorem 2.1.** Let  $[a, b]$  be the interval of integration and  $n$  be the number of subintervals within the interval  $[a, b]$ . Assume  $f$  is continuous over  $[a, b]$ , and let  $\zeta_0 \in [a, b]$  be such that

$$|f(\zeta_0)| = \max_{a \leq \zeta \leq b} |f(\zeta)|.$$

Then the error,  $E_T$ , for the tangent line approximation satisfies

$$|E_T| \leq \frac{(b-a)^3}{6n^2} |f''(\zeta_0)|.$$

The remainder of this section will prove Theorem 2.1.

Assume the function  $f$  is twice differentiable on  $(a, b)$ . To find the total error for the tangent line approximation, the sum of the errors for each interval must be calculated [2]. Thus we have the total error,  $E_T$ , given by

$$E_T = \sum_{i=0}^{n-1} E_i,$$

where  $E_i$  is the error for the  $i$ th subinterval.

Initially, we found the formula for the approximation by integrating each tangent line of the form

$$T_i(x) = f'(x_i)(x - x_i) + f(x_i)$$

(where  $x_i$  is the left endpoint of the  $i$ th subinterval) and summing over each interval. Within this calculation, however, lies a degree of inaccuracy, especially when  $n$  (the number of subintervals) is relatively small. To quantify this inaccuracy, let's write

$$f(x) = T_i(x) + R_i(x)$$

so that  $R_i(x)$  is the remainder in the first-degree Taylor approximation of  $f$ . Then  $R_i(x)$  satisfies

$$R_i(x) = \frac{f''(\zeta_i)}{2!} (x - x_i)^2$$

for some  $\zeta_i \in [x_i, x]$  (where  $\zeta_i$  depends on  $x$ ) [1, p. 3].

Assume  $f''$  is continuous on an interval  $[a, b]$ . Then by the Intermediate Value Theorem [1, p. 7], there exists  $\zeta_0 \in [a, b]$  such that

$$|f''(\zeta_0)| = \max_{a \leq \zeta \leq b} |f''(\zeta)| \geq |f''(\zeta_i)|, \quad i = 0, 1, 2, \dots, n-1.$$

Therefore

$$|R_i(x)| \leq \frac{|f''(\zeta_0)|}{2!} (x - x_i)^2, \quad i = 0, 1, 2, \dots, n-1.$$



Thus, we find the error,  $E_i$ , over a single interval  $[x_i, x_{i+1}]$  by integrating the above inequality over the subinterval:

$$\begin{aligned} |E_i| &= \left| \int_{x_i}^{x_{i+1}} (f(x) - T_i(x)) dx \right| \leq \int_{x_i}^{x_{i+1}} |f(x) - T_i(x)| dx \\ &= \int_{x_i}^{x_{i+1}} |R_i(x)| dx \\ &\leq \frac{|f''(\zeta_0)|}{2!} \int_{x_i}^{x_{i+1}} (x - x_i)^2 dx \\ &= \frac{|f''(\zeta_0)|}{6} (x_{i+1} - x_i)^3. \end{aligned}$$

Thus, the error for one subinterval  $[x_i, x_{i+1}]$  is bounded by

$$|E_i| \leq \frac{\Delta x^3}{6} |f''(\zeta_0)|,$$

where  $\Delta x = x_{i+1} - x_i$ .

The total error is simply the sum of the errors from each segment. And so the total error in the tangent approximation is

$$|E_T| = \left| \sum_{i=0}^{n-1} E_i \right| \leq \sum_{i=0}^{n-1} \frac{\Delta x^3}{6} |f''(\zeta_0)| \leq \frac{(b-a)^3}{6n^2} |f''(\zeta_0)|.$$

## 2.3 Comparison with Other Methods

The error given in Theorem 2.1 decreases as  $n$  gets larger, and the approximation becomes more exact as  $n \rightarrow \infty$ . When choosing the best method to approximate the definite integral, a few factors must be taken into consideration. In order to apply the tangent line approximation, the function must be differentiable, since we must take the derivative in the computation. We saw earlier that simply substituting the approximation of the derivative into the derivative itself gives us the approximation for the Trapezoidal Rule.

The error for the Trapezoidal Rule is given in a similar form as that in Theorem 2.1:

$$|E_{Tr}| \leq \frac{(b-a)^3}{12n^2} |f''(\zeta_0)| \quad [2]. \quad (2.3)$$

Notice the only difference between the two equations is in the denominator. Since the denominator in (2.3) is twice as large, the Trapezoidal Rule in general approximates the definite integral more efficiently than the tangent line approximation. With this realization in mind, it would be worthwhile to ask if we could improve the accuracy of the tangent line approximation. For example, what if instead of calculating the tangent line at the left endpoint of each subinterval we calculated the tangent line at the midpoint?

This approximation using the midpoint is over the same interval  $[a, b]$  with the subintervals  $a = x_0, x_1, x_2, \dots, x_n = b$ . Here we take the midpoint,  $\bar{x}_i$ , of each subinterval  $[x_i, x_{i+1}]$ , and derive a formula in a similar way as (2.1), simply substituting the left endpoint  $x_i$  with  $\bar{x}_i$  in the original equation of our tangent line over each subinterval. Thus, we have

$$\begin{aligned} & \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[ f'(\bar{x}_i)(x - \bar{x}_i) + f(\bar{x}_i) \right] dx \\ &= \sum_{i=0}^{n-1} \left[ f'(\bar{x}_i) \left( \frac{1}{2}(x_{i+1}^2 - x_i^2) - (\bar{x}_i)\Delta x \right) + f(\bar{x}_i)\Delta x \right]. \end{aligned}$$

Substituting

$$\bar{x}_i = \frac{1}{2}(x_{i+1} + x_i)$$

$$\Delta x = x_{i+1} - x_i$$

and

$$x_{i+1}^2 - x_i^2 = (x_{i+1} + x_i)(x_{i+1} - x_i)$$

gives

$$\begin{aligned} & \sum_{i=0}^{n-1} \left[ f'(\bar{x}_i) \left( \frac{1}{2}(x_{i+1} + x_i)(x_{i+1} - x_i) - \left( \frac{1}{2}(x_{i+1} + x_i) \right) (x_{i+1} - x_i) \right) + f(\bar{x}_i)\Delta x \right] \\ &= \sum_{i=0}^{n-1} f(\bar{x}_i)\Delta x. \end{aligned}$$

This is equivalent to the Midpoint Rule given in (1.1). Note from geometry that given a value  $f(\bar{x}_i)$ , the area under any straight line passing through  $f(\bar{x}_i)$  over that subinterval will be the same. This is illustrated in Figure 3. The Midpoint Rule is the particular case when the line is horizontal. Therefore, altering the tangent line approximation to use the midpoints of the subintervals instead of the left endpoint does not yield a new approximation. However, it does improve accuracy, since the error formula for the Midpoint Rule is given by

$$|E_M| \leq \frac{(b-a)^3}{24n^2} |f''(\zeta)| \quad [3, \text{p. 534}].$$

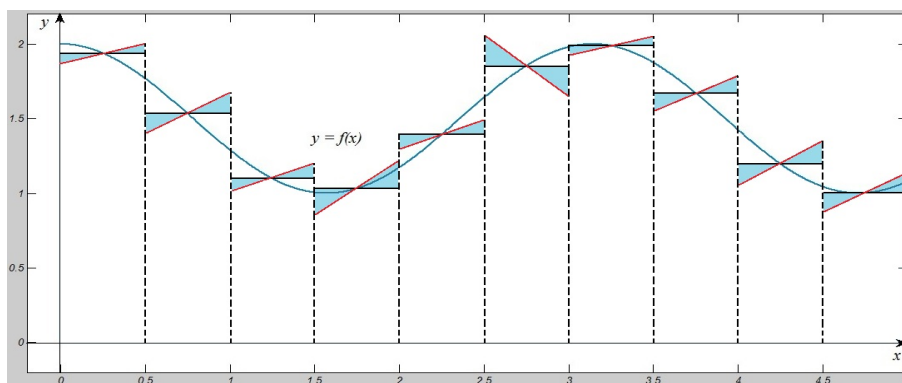


Figure 3: Given any two lines passing through the midpoint  $(\bar{x}_i, f(\bar{x}_i))$  of a subinterval, the area beneath each line will be the same. Note here that the shaded area above and below the horizontal line is the same for each subinterval.

The difference between the errors for the Midpoint Rule and the tangent line approximation is, once again, the denominator. The denominator for the Midpoint Rule is four times larger than that for the tangent line approximation, indicating that, in general, the Midpoint Rule will give a more accurate approximation.

Although the tangent line approximation seems to be less efficient compared with the Midpoint and Trapezoidal Rules, repeated examples have illustrated that this new approximation performs better than the left and right endpoint approximations. This is a reasonable conclusion, since both the left and right endpoint approximations tend to significantly under-approximate and over-approximate the area underneath the curve for a positive function compared with the other methods mentioned. However, the accuracy and dependence of each approximation depends on the given function to which we are applying our approximation.

## 2.4 Applying the Tangent Line Approximation

Let's manually apply the tangent line approximation to approximate the area beneath the curve  $f(x) = \sin^2(x)$ , from  $x = 1$  to  $x = 5$  with  $n = 4$  subintervals.

Here, we have  $\Delta x = \frac{5-1}{4} = 1$ , and  $f'(x) = 2 \sin(x) \cos(x)$ .

Applying the formula derived in (2.1), we obtain

$$\begin{aligned}
 \int_1^5 f(x)dx &\simeq \sum_{i=0}^3 \left[ \Delta x \left( \frac{1}{2} f'(x_i) \Delta x + f(x_i) \right) \right] \\
 &= \sum_{i=0}^3 \left[ \frac{1}{2} f'(x_i) + f(x_i) \right] \\
 &= \left[ \frac{1}{2} (2 \sin(1) \cos(1)) + \sin^2(1) \right] + \left[ \frac{1}{2} (2 \sin(2) \cos(2)) + \sin^2(2) \right] \\
 &+ \left[ \frac{1}{2} (2 \sin(3) \cos(3)) + \sin^2(3) \right] + \left[ \frac{1}{2} (2 \sin(4) \cos(4)) + \sin^2(4) \right] \\
 &\approx 2.558778942 \text{ rounding to nine decimal places.}
 \end{aligned}$$

We can compare this approximation to the exact value of the definite integral:

$$\int_1^5 \sin^2(x)dx = \frac{1}{2} \left[ x - \frac{1}{2} \sin(2x) \right] \Big|_1^5 = 2.363329634$$

rounding to 9 decimal places.

Note that we chose  $n$  to be relatively small in order to simplify the computational demonstration of the method. As  $n$  grows larger, the accuracy of the approximation increases substantially. Below, we show the result of the tangent line approximation compared with the Midpoint Rule, Trapezoidal Rule, and Left Endpoint Approximation, as  $n$  increases.

$n$ -Subintervals	Tangent Line	Midpoint	Trapezoidal	Left Endpoint
$n = 10$	2.3995948697	2.3732023841	2.3437421589	2.3014496896
$n = 50$	2.3648614575	2.3637174756	2.3625542003	2.3540957064
$n = 100$	2.3637149095	2.3634265404	2.3631358380	2.3589065910
$n = 500$	2.3633451184	2.3633335100	2.3633218834	2.3624760340

The above calculations were quickly executed using the mathematical software Matlab. Note as  $n$  increases, each approximation increases in accuracy. In particular, the tangent line approximation rapidly decreases in error as the number of subintervals increases. This rapid decrease in error is illustrated in Figure 4 that follows.

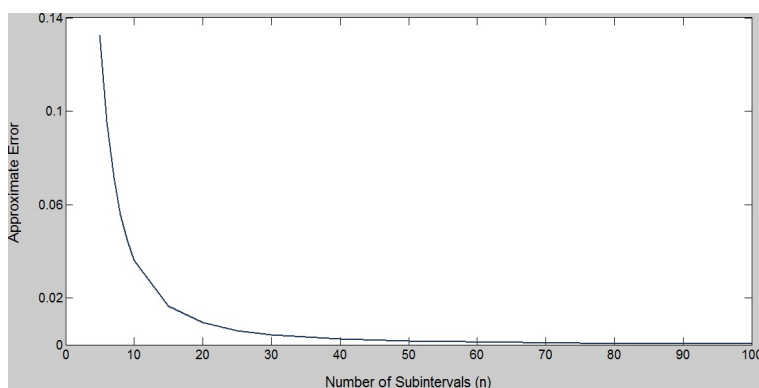


Figure 4: Graph of the approximate error for the tangent line approximation as the number of subintervals  $n$  increases.

### 3 Tangent Plane Approximation of Definite Integrals and Error Formula

#### 3.1 Introduction to Volumes and Double Integrals

We will now extend the ideas of Section 2 to integrals of functions of two variables. We consider a function  $f$  of two variables defined on a closed region  $R = [a, b] \times [c, d]$  with  $(x, y)$  in the two-dimensional space of real numbers such that  $a \leq x \leq b$  and  $c \leq y \leq d$ .

The graph of  $f$  is a surface with equation  $z = f(x, y)$  (illustrated by Figure 5 below). The region  $R$  is the projection of the surface onto the  $xy$  plane. We wish to find the volume of the surface  $S$  which lies above  $R$  and below the graph of the positive function  $f$  [3, p. 998].

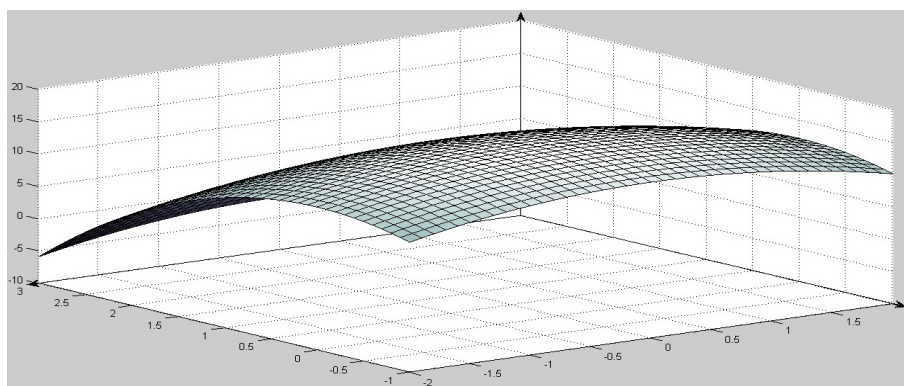


Figure 5: Three-dimensional graph of a surface  $f(x, y)$ .

To find this volume, we divide the rectangle  $R$  into subrectangles. We divide the interval  $[a, b]$  into  $m$  subrectangles  $[x_i, x_{i+1}]$  of equal width  $\Delta x = \frac{b-a}{m}$  and divide  $[c, d]$  into  $n$  subrectangles  $[y_j, y_{j+1}]$  of equal width  $\Delta y = \frac{d-c}{n}$ .

And so, we find the volume beneath  $f$  by double integrating over the region  $R$  as follows, for sample points  $(x_{ij}^*, y_{ij}^*)$  in

$$[x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

and  $\Delta A = \Delta x \Delta y$ ,

$$\int \int_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A \quad [3, \text{p. } 1000].$$

### 3.2 Tangent Plane Approximation

The tangent plane method of approximation uses tangent planes to approximate the given function  $f$  of two variables over each subrectangle by choosing the lower left point of each rectangle and finding the linear approximation/tangent plane of  $f$  at that point. This plane is given by

$$T_{ij}(x, y) = f_x(x_i, y_j)(x - x_i) + f_y(x_i, y_j)(y - y_j) + f(x_i, y_j) \quad [3, \text{p. } 941].$$

Here, we let  $x_i = x_0 + i\Delta x$  and  $y_j = y_0 + j\Delta y$ , where  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ .

The value of the integral for one subrectangle is given by

$$\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} T_{ij}(x, y) dx dy.$$

Therefore, in order to find the combined value over all the subrectangles, we must sum over the region  $R$ . This gives us our tangent plane approximation of definite integrals:

$$\begin{aligned} & \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \left[ f_x(x_i, y_j)(x - x_i) + f_y(x_i, y_j)(y - y_j) + f(x_i, y_j) \right] dx dy \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left[ f_x(x_i, y_j) \left[ \frac{1}{2}(x_i + \Delta x)^2 - (x_i)(x_i + \Delta x) + \frac{1}{2}(x_i)^2 \right] \Delta y \right. \\ & \quad \left. + f_y(x_i, y_j) \left[ \frac{1}{2}(y_j + \Delta y)^2 - (y_j)(y_j + \Delta y) + \frac{1}{2}(y_j)^2 \right] \Delta x + f(x_i, y_j) \Delta x \Delta y \right]. \end{aligned}$$

Substituting  $x_{i+1} - x_i = \Delta x$ ,  $y_{j+1} - y_j = \Delta y$ ,  $x_{i+1} = x_i + \Delta x$ , and  $y_{j+1} = y_j + \Delta y$ , we obtain

$$\int_a^b \int_c^d f(x, y) dy dx \approx \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Delta x \Delta y \left[ f_x(x_i, y_j) \frac{1}{2} \Delta x + f_y(x_i, y_j) \frac{1}{2} \Delta y + f(x_i, y_j) \right]. \quad (3.1)$$

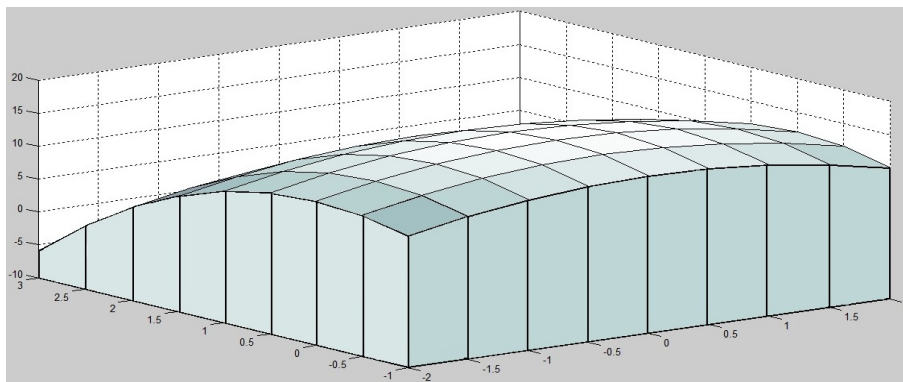


Figure 6: Approximation of the integral  $\int_a^b \int_c^d f(x, y) dy dx$  using the tangent plane approximation, and the volume of solids below tangent planes at each subrectangle.

Remember that the tangent plane approximation above uses tangent planes found at the lower left corner of each subrectangle. If, instead, we choose to find the tangent plane at the midpoint of each subrectangle, we obtain the following familiar Midpoint Rule for double integrals:

$$\iint_R f(x, y) \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A.$$

Note from geometry that given the value  $f(\bar{x}_i, \bar{y}_j)$  at the midpoint, the volume under any plane passing through  $f(\bar{x}_i, \bar{y}_j)$  over the subrectangle  $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$  will be the same. The Midpoint Rule is the particular case when the plane is horizontal and will be equivalent to the tangent plane approximation at the midpoint.

Remark: Similar to the tangent line approximation, the limitation to using the tangent plane approximation in calculations resides in the fact that in order to use (3.1) to approximate definite integrals, it is necessary that the given function be differentiable. Taking this limitation into consideration, we can derive a similar formula to the tangent plane approximation by using the approximations of the partial derivatives. Thus, we have the following substitutions:

$$f_x(x_i, y_j) \simeq \frac{f(x_i + \Delta x, y_j) - f(x_i, y_j)}{\Delta x}$$

and

$$f_y(x_i, y_j) \simeq \frac{f(x_i, y_j + \Delta y) - f(x_i, y_j)}{\Delta y} \quad [3, \text{p. 926}].$$

Substituting these equations into (3.1) gives

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dy dx &\simeq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Delta x \Delta y \left[ \left( \frac{f(x_i + \Delta x, y_j) - f(x_i, y_j)}{\Delta x} \right) \frac{1}{2} \Delta x \right. \\ &\quad \left. + \left( \frac{f(x_i, y_j + \Delta y) - f(x_i, y_j)}{\Delta y} \right) \frac{1}{2} \Delta y + f(x_i, y_j) \right] \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Delta x \Delta y \frac{1}{2} [f(x_{i+1}, y_j) + f(x_i, y_{j+1})]. \end{aligned}$$

Note that when we approximate the derivative, as above, we obtain the three-dimensional Trapezoidal Rule. We are taking the height for a particular subrectangle to be the average of the heights at two corners (similar to the two-dimensional Trapezoidal Rule which takes the height to be the average of the two endpoints).

### 3.3 Error Formula for Tangent Plane Approximation

The error formula for the tangent plane approximation can be derived using similar methods that were used to derive the error formula for the tangent line approximation. We can find the total error for the three-dimensional tangent plane approximation by taking the sum of the errors for each subrectangle. Therefore, the total error of the approximation,  $E_{TP}$ , is given by

$$E_{TP} = \sum_{i=1}^n \sum_{j=1}^m E_{ij}$$

where  $E_{ij}$  is the error for the  $(i, j)$ th subrectangle.

The formula for the tangent plane approximation was initially found by double integrating the formula of the tangent plane and summing over all the subrectangles. As with the tangent line approximation, the tangent plane approximation contains a degree of inaccuracy, especially when the number of subrectangles ( $n$  and  $m$ ) is relatively small.

The error bound for the tangent plane approximation,  $E_{TP}$ , is given in the following theorem.



**Theorem 3.1.** *Let  $[a, b] \times [c, d]$  be the region of integration. We divide the interval  $[a, b]$  into  $m$  subrectangles of equal width and divide  $[c, d]$  into  $n$  subrectangles of equal width. Suppose that  $M$  is an upper bound on  $|f_{xx}|$ ,  $|f_{xy}|$ , and  $|f_{yy}|$  over the region  $[a, b] \times [c, d]$ , and  $f_{xx}$ ,  $f_{xy}$ , and  $f_{yy}$  are continuous on  $[a, b] \times [c, d]$ . Then the error for the tangent plane approximation is given by*

$$|E_{TP}| \leq M \left[ \frac{(b-a)^3(d-c)}{6n^2} + \frac{(b-a)^2(d-c)^2}{4nm} + \frac{(b-a)(d-c)^3}{6m^2} \right].$$

The remainder of this section will prove Theorem 3.1.

We let

$$T_{ij}(x, y) = f_x(x_i, y_j)(x - x_i) + f_y(x_i, y_j)(y - y_j) + f(x_i, y_j)$$

be the first degree Taylor polynomial in two variables of  $f$ , and  $R_{ij}$  the remainder (error) of the Taylor series on the subrectangle  $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$  such that

$$f(x, y) = T_{ij}(x, y) + R_{ij}(x, y) \quad [1, \text{p. } 4].$$

Assume  $f_{xx}$ ,  $f_{xy}$ , and  $f_{yy}$  are continuous over the region  $R = [a, b] \times [c, d]$ , and suppose that  $M$  is an upper bound on  $|f_{xx}|$ ,  $|f_{xy}|$ , and  $|f_{yy}|$  on  $R$ . Then the remainder  $R_{ij}$  satisfies the inequality

$$|R_{ij}(x, y)| \leq \frac{M}{2!} \left[ (x - x_i)^2 + 2|(x - x_i)(y - y_j)| + (y - y_j)^2 \right]. \quad (3.2)$$

We calculate the error over a single subrectangle  $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$  by double integrating the error  $R_{ij}$  over the subrectangle. And so, we have that the error for each subrectangle is

$$\begin{aligned} |E_{ij}| &= \left| \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} R_{ij}(x, y) dx dy \right| \\ &\leq \frac{M}{2!} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \left[ (x - x_i)^2 + 2(x - x_i)(y - y_j) + (y - y_j)^2 \right] dx dy \\ &= \frac{M}{2} \left[ \frac{1}{3}(x_{i+1} - x_i)^3(y_{j+1} - y_j) + \frac{1}{2}(x_{i+1} - x_i)^2(y_{j+1} - y_j)^2 + \frac{1}{3}(x_{i+1} - x_i)(y_{j+1} - y_j)^3 \right] \\ &= M \left[ \frac{1}{6}(\Delta x)^3(\Delta y) + \frac{1}{4}(\Delta x)^2(\Delta y)^2 + \frac{1}{6}(\Delta x)(\Delta y)^3 \right] \end{aligned}$$

where  $x_{i+1} - x_i = \Delta x$  and  $y_{j+1} - y_j = \Delta y$ . Note that the absolute value on the right hand side of (3.2) is removed during integration, since we are only considering values of  $x$  and  $y$  that are greater than  $x_i$  and  $y_j$ , respectively.

We find the total error by summing over all subrectangles. So, the total error for the tangent plane approximation is

$$\begin{aligned} |E_{TP}| &\leq \sum_{i=1}^n \sum_{j=1}^m M \left[ \frac{1}{6}(\Delta x)^3(\Delta y) + \frac{1}{4}(\Delta x)^2(\Delta y)^2 + \frac{1}{6}(\Delta x)(\Delta y)^3 \right] \\ &\leq M \left[ \frac{nm}{6}(\Delta x)^3(\Delta y) + \frac{nm}{4}(\Delta x)^2(\Delta y)^2 + \frac{nm}{6}(\Delta x)(\Delta y)^3 \right] \\ &\leq M \left[ \frac{(b-a)^3(d-c)}{6n^2} + \frac{(b-a)^2(d-c)^2}{4nm} + \frac{(b-a)(d-c)^3}{6m^2} \right] \end{aligned}$$

where  $\Delta x = \frac{b-a}{n}$  and  $\Delta y = \frac{d-c}{m}$ .

### 3.4 Applying the Tangent Plane Approximation

Let's manually apply the tangent plane approximation to approximate the volume beneath the surface  $f(x, y) = x - 3y^2$ , where  $0 \leq x \leq 2$  and  $1 \leq y \leq 2$ . For a simplified application of the method, assign subrectangles  $m$  and  $n$  such that  $m = n = 2$ .

We have that  $\Delta x = \frac{b-a}{m} = \frac{2-0}{2} = 1$  and  $\Delta y = \frac{d-c}{n} = \frac{2-1}{2} = \frac{1}{2}$ .

The partial derivatives necessary for the computation are  $f_x(x, y) = 1$  and  $f_y(x, y) = -6y$ .

Applying the formula derived in (3.1), we obtain

$$\begin{aligned} \int_0^2 \int_1^2 f(x, y) dy dx &\simeq \sum_{i=0}^1 \sum_{j=0}^1 \Delta x \Delta y \left[ f_x(x_i, y_j) \frac{1}{2} \Delta x + f_y(x_i, y_j) \frac{1}{2} \Delta y + f(x_i, y_j) \right] \\ &= (1) \left( \frac{1}{2} \right) \left[ f_x(0, 1) \left( \frac{1}{2} \right) (1) + f_y(0, 1) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) + f(0, 1) \right] + (1) \left( \frac{1}{2} \right) \left[ f_x \left( 0, \frac{3}{2} \right) \left( \frac{1}{2} \right) (1) \right. \\ &+ \left. f_y \left( 0, \frac{3}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) + f \left( 0, \frac{3}{2} \right) \right] + (1) \left( \frac{1}{2} \right) \left[ f_x(1, 1) \left( \frac{1}{2} \right) (1) + f_y(1, 1) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right. \\ &+ \left. f(1, 1) \right] + (1) \left( \frac{1}{2} \right) \left[ f_x \left( 1, \frac{3}{2} \right) \left( \frac{1}{2} \right) (1) + f_y \left( 1, \frac{3}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) + f \left( 1, \frac{3}{2} \right) \right] \\ &= -11.50. \end{aligned}$$

We have that

$$\int_0^2 \int_1^2 (x - 3y^2) dy dx = -12.$$

Therefore, the error in our calculation is  $|-12 - (-11.50)| = 0.50$ .

We chose  $m$  and  $n$  to be very small in order to simplify the computational demonstration of the method. As the number of subrectangles  $m$  and  $n$  grow large, the error decreases

substantially. Below, we show the result of the tangent plane approximation compared with the Midpoint Rule and Trapezoidal Rule (both three-dimensional), as  $m$  and  $n$  increase substantially.

nm-Subrectangles	Tangent Plane	Midpoint	Trapezoidal
$m = n = 10$	-11.980000000	-11.994999999	-12.010000000
$m = n = 50$	-11.999200000	-11.999800000	-12.000400000
$m = n = 100$	-11.999800000	-11.999950000	-12.000100000

The above calculations were quickly executed using the mathematical software Matlab. Note as  $m$  and  $n$  increase, each approximation increases in accuracy. In particular, the tangent plane approximation rapidly decreases in error as the number of subrectangles is maximized. This rapid decrease in error is illustrated in Figure 7 that follows.

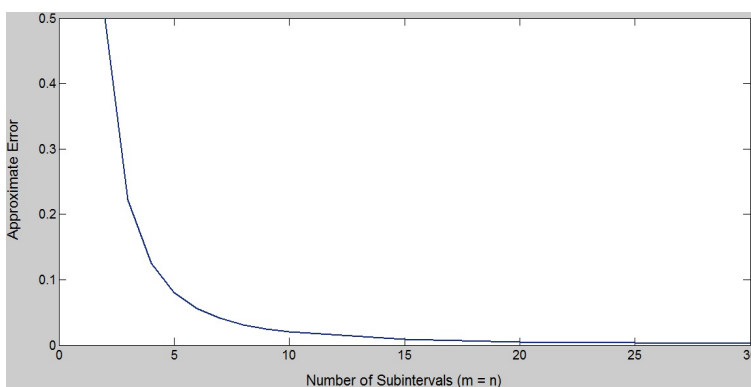


Figure 7: Graph of the approximate error for the tangent plane approximation as the number of subrectangles  $m$  and  $n$  increases.

## 4 Conclusion

In the preceding sections, we derived two alternate methods for approximating definite integrals in two and three-dimensional spaces, respectively. The tangent line approximation involves calculating the tangent lines at the left endpoints of subintervals over the region of integration. For a positive function, the formula is essentially the result of summing the areas beneath each tangent line corresponding to each subinterval. In a similar fashion, the tangent plane approximation involves calculating the tangent planes at the lower left corners of subrectangles over the rectangular region of integration. Figuratively, for a positive function of two variables, the formula is the result of summing the volumes beneath each tangent plane corresponding to each subrectangle.

## 5 Future Work

Note that both the tangent line and tangent plane methods can be extended to other Taylor approximations in hopes of decreasing the error and increasing the accuracy. For the tangent line approximation, we approximated the function  $f$  using the equation of the tangent line, which is the first degree Taylor polynomial of  $f$ . We then used this equation to derive a formula for approximating definite integrals. We can extend this idea to approximating the function  $f$  using polynomials of larger degrees instead of simply using tangent lines. Therefore, we could approximate the given curve using Taylor polynomials of  $f$ , and then follow a similar procedure for deriving a definite integral approximation. For instance, if we want to approximate  $f$  using a polynomial of degree  $n$ , we would use the  $n$ th degree Taylor polynomial of  $f$ , given that  $f$  is at least  $n$  times differentiable.

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