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# Linearization and Stability Analysis of Nonlinear Problems

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**Abstract.** The focus of this paper is on the use of linearization techniques and linear differential equation theory to analyze nonlinear differential equations. Often, mathematical models of real-world phenomena are formulated in terms of systems of nonlinear differential equations, which can be difficult to solve explicitly. To overcome this barrier, we take a qualitative approach to the analysis of solutions to nonlinear systems by making phase portraits and using stability analysis. We demonstrate these techniques in the analysis of two systems of nonlinear differential equations. Both of these models are originally motivated by population models in biology when solutions are required to be non-negative, but the ODEs can be understood outside of this traditional scope of population models. In fact, allowing solutions for these equations to be negative provides some very interesting mathematical problems, and demonstrates the utility of the analysis techniques to be described in this article. We provide stability analysis, phase portraits, and numerical solutions for these models that characterize behaviors of solutions based only on the parameters used in the formulation of the systems. The first part of this paper gives a survey of standard linearization techniques in ODE theory. The second part of the paper presents applications of these techniques to particular systems of nonlinear ODEs, which includes some original results by extending the analysis of solutions lying anywhere in the plane $\mathbb{R}^2$, rather than only those in the first quadrant.

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# 1 Introduction

The focus of this paper is on the use of linearization techniques and linear differential equation theory to analyze nonlinear differential equations. Often, mathematical models of real-world phenomena are formulated in terms of systems of nonlinear differential equations, which can be difficult to solve explicitly. To overcome this barrier, we take a qualitative approach to the analysis of solutions to nonlinear systems by making phase portraits and using stability analysis. We demonstrate these techniques in the analysis of two systems of nonlinear differential equations. Both of these models are originally motivated by population models in biology when solutions are required to be non-negative, but the ordinary differential equations (ODEs) can be understood outside of this traditional scope. In fact, allowing solutions for these equation to be negative provides some very interesting mathematical problems, and demonstrates the utility of the analysis techniques to be described in this article.

Generally, given any system of first-order ODEs (possibly nonlinear), we want to create a qualitative characterization of the behavior of solutions depending on their initial conditions. We do this by first finding equilibrium solutions, and then using stability analysis to make a phase portrait for the system that indicates the general behavior of solutions depending on their initial conditions. The stability analysis carried out at each equilibrium involves linearization techniques and solution methods for systems of linear ODEs, which further involve ideas from linear algebra, including computing eigenvalues and eigenvectors, among other topics. There are some foundational ideas that we will not discuss in depth so that we can move more quickly to the focus of our work. The theory behind these topics is crucial to our work, but our analysis begins where the theory ends.

Although most of the theory discussed in this paper can be extended to arbitrary dimensions, for simplicity we will restrict to analysis in two dimensions. For more information on the extension to arbitrary dimensions, see for example the text by Boyce and DiPrima [BD]. We further remark that although we limit our analysis to ODEs, linearization techniques also play a role in the solution of partial differential equations (PDEs). Though there are differences between the application of linearization techniques to ODEs and PDEs, the ideas of simplifying the problem at hand and analyzing eigenvalues to gain a general understanding of the solutions are shared. This can be observed, for instance, in separation of variable techniques applied to solve the heat equation and the wave equation. See the text by Haberman [H] for examples, and the references therein for more information.

Our article is organized as follows. In Section 2, we establish some general notation, formulate ODE problems, and discuss the pertinent existence and uniqueness theory. In Section 3, we discuss explicitly solving systems of linear ODEs and linearization of nonlinear systems of ODEs. Section 4 is used to define and describe stability of linear and nonlinear systems. Finally, in Section 5, we use the linearization and stability analysis described in the previous sections to analyze two systems of ODEs, which includes phase portraits, stability analysis of equilibrium solutions, and numerical simulations.

All graphs were generated in Matlab using open source software written by J. C. Polking and J. Castellanos [PC], and available `http://math.rice.edu/~dfield/dfpp.html`.

# 2 Systems of Ordinary Differential Equations

An autonomous first order system of differential equations is of the form

$$\begin{aligned} x_1'(t) &= F_1(x_1(t), x_2(t)), \\ x_2'(t) &= F_2(x_1(t), x_2(t)), \end{aligned} \tag{1}$$

or simply $x' = F(x)$, where $F(x) = (F_1(x), F_2(x))$ is defined for $x = (x_1, x_2) \in \mathbb{R}^2$. Here the variables $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are all parameterized by a single, independent variable $t \in \mathbb{R}$, typically chosen to represent time. A solution to the system of ODEs is of the form $x = \phi(t)$, where

$$x_1 = \phi_1(t),\ x_2 = \phi_2(t),$$

each function $\phi_j$ is differentiable and $\phi_j'(t) = F_j(\phi_1(t), \phi_2(t))$ for $j = 1, 2$. In this work, we will only work with autonomous systems of ODEs. There are non-autonomous systems of ODEs as well, which are formulated by allowing the functions $F_1, F_2$ to depend on $t$ and $x_1, x_2$, but this is outside the scope of our analysis.

For a given system of ODEs of the form (1), an initial value problem (IVP) associated with the system is formulated by the equations in (1) combined with an initial condition $x(0) = x^0$ for some fixed $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$.

## 2.1 Existence and Uniqueness of Solutions

Before analyzing systems of ordinary differential equations, we should first establish that the ODE has a solution for us to further analyze. This is the purpose of the existence and uniqueness theory for systems of differential equations. The appropriate existence and uniqueness result for our setting is the following theorem, which can be found as stated here in the text by Boyce and DiPrima [BD].

**Theorem 1.** *Let each of the functions $F_j$ for $j = 1, 2$ as well as their partial derivatives $\partial F_j/\partial x_k$ for $j = 1, 2$ and $k = 1, 2$ be continuous in a region $R$ of $t, x_1, x_2$ space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, $\alpha_2 < x_2 < \beta_2$, and let the point $(t_0, x_1^0, x_2^0)$ be in $R$. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution $x_1 = \phi_1(t), x_2 = \phi_2(t)$ of the system of differential equations that also satisfies the initial condition $x(t_0) = (x_1^0, x_2^0)$.*

While more general versions of this theorem do exist, for our purposes it is enough to guarantee that the systems we work with have unique solutions. Thus, we've assured the existence and uniqueness of solutions to our systems of ODEs. So we can move forward with our analysis of systems of ordinary differential equations.

## 2.2 Linear Systems of ODEs

Linear systems of first-order ordinary differential equations are of the form $x' = F(x)$, where each of the functions $F_1, F_2$ describe linear relationships between $x_1, x_2$. In this scenario, we write $x' = F(x)$ as $x' = Ax$ where $A$ is a matrix such that each entry $a_{ij}$ corresponds to the coefficient of $x_j$ in the function $F_i$; for this reason $A$ is referred to as the coefficient matrix of the linear system. Thus, linear systems can be written as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}, \tag{2}$$

where $a_{ij}$ is the $i^{th}$ row and $j^{th}$ column entry of $A$. If one were to allow the matrix $A$ to depend on $t$, the system would be a non-autonomous linear system of ODEs, which again we will not address in this paper.

## 2.3 Nonlinear Systems of ODEs

In the last section, we described systems of linear differential equations, but many of the problems in ODE theory come from nonlinear equations. If the variables in the equations from (1) do not exhibit linear relationships, then the system is called nonlinear. As we will see, nonlinear systems can be very complicated, if not impossible, to solve explicitly; however, when it comes to modeling real-world phenomena, the majority of the systems that arise are nonlinear. To be able to analyze these systems we will linearize them at their equilibria, and then construct phase portraits to visualize the trajectories of the solutions to the system. It is in this way that we can overcome the barrier of not being able to solve nonlinear systems explicitly. The remainder of this paper will discuss this process in great detail.

## 2.4 Equilibria

A relatively simple aspect of systems of ODEs that can be observed without a great deal of analysis is the locations of the equilibria. Equilibria for a system of the form (1), which are sometimes also referred to as critical points of the system, are points $X \in \mathbb{R}^2$ where

$$F_1(X) = F_2(X) = 0.$$

It is easy to notice that if $X \in \mathbb{R}^2$ is an equilibrium solution of the system (1), then the constant functions $x_1(t) = X_1,\ x_2(t) = X_2$ define a solution $x(t) = (X_1, X_2)$ to the system of equations (hence the name equilibrium solution). Finding these equilibrium solutions corresponds to solving the nonlinear algebraic equations $F(x) = 0$. This process starts to create a picture of the behavior of solutions to a system like (1).

## 2.5 Stability

The majority of our analysis of systems of ODEs will focus on whether or not the systems have stable equilibria. We characterize an equilibrium as stable or unstable based on the behavior of solutions whose initial conditions are in the neighborhood of the equilibrium. If solutions near a critical point of a system stay close to the critical point as $t$ approaches infinity, we think of the critical point as stable; if this condition is not met then the critical point is unstable. Furthermore, we call a stable critical point asymptotically stable if, over time, the solutions actually approach the critical point as opposed to simply staying within a certain radius. More precisely, we give the following standard definition that can be found in the text by Boyce and DiPrima [BD]:

**Definition 1.** Let $X \in \mathbb{R}^2$ be a critical point of a system of ODEs of the form (1).

- The critical point $X$ is stable if, for any $\epsilon > 0$, there is a $\delta > 0$ such that if a solution $x = \phi(t)$ satisfies $||\phi(0) - X|| < \delta$, then

  $$||\phi(t) - X|| < \epsilon \text{ for all } t > 0.$$

  Here $||x|| = \sqrt{x_1^2 + x_2^2}$ denotes the Euclidean norm on $\mathbb{R}^2$.

- The critical point $X$ is unstable if it is not stable as defined above.

- The critical point $X$ is asymptotically stable if there exists a $\delta > 0$ such that if a solution $x = \phi(t)$ satisfies $||\phi(0) - X|| < \delta$, then

  $$\lim_{t\to\infty} \phi(t) = X.$$

Note that in our definition of stable equilibrium and asymptotically stable equilibrium, we implicitly assume that the solution $\phi(t)$ exists for all time $t > 0$. The existence of a solution for all $t > 0$ given an initial condition $x(0) = x^0$ is not guaranteed to us by the existence and uniqueness theorem stated earlier in this section; the theorem only assures us that a solution exists locally in time. For us, the existence of $\phi(t)$ for all time $t > 0$ for solutions near a stable or asymptotically stable equilibrium is a matter of definition; if an equilibrium point $X$ is stable (or asymptotically stable), then there must be a $\delta > 0$ for which any solution $\phi(t)$, satisfying $||\phi(0) - X|| < \delta$, must exist for all time in addition to the stability conditions above. One can delve further into the global existence theory of systems of ODEs to develop a more complete picture of existence of solutions, but for our purposes the above definition is sufficient and going deeper into the general theory would detract from the main purpose of this article. So we leave our discussion of stability, existence, and uniqueness at that. With our definitions now set, we can now begin to look at systems of ODEs and classify their equilibria based on the behaviors of solutions.

# 3 Methods of Analysis

Our main goal when analyzing systems of ordinary differential equations is to gain an understanding of the behaviors of the solutions to the systems. The natural approach for analyzing a system is to solve it explicitly, and this method works well if the system is linear. If the system is not linear, then solving explicitly can be very complicated (maybe impossible), so we instead linearize the system at its equilibria and gain a qualitative understanding of the solutions by analyzing the linearized system.

## 3.1 Solving Explicitly

To demonstrate how to solve a linear system of ODEs and to introduce some general terminology about types of equilibria, we will work through some examples of linear systems in two dimensions. Let's begin with the simple system

$$x' = \underbrace{\begin{pmatrix} 1 & 1 \\ 16 & 1 \end{pmatrix}}_{A} x. \tag{3}$$

It can be shown that solutions to (3) are of the form

$$x = c_1\xi^{(1)}e^{\lambda_1 t} + c_2\xi^{(2)}e^{\lambda_2 t} = \begin{pmatrix} c_1e^{5t} + c_2e^{-3t} \\ 4c_1e^{5t} - 4c_2e^{-3t} \end{pmatrix}, \tag{4}$$

where $\lambda_1 = 5$ and $\lambda_2 = -3$ are the eigenvalues of $A$, $\xi^{(1)} = (1, 4)$ and $\xi^{(2)} = (1, -4)$ are the eigenvectors of $A$, and $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. See the text by Boyce and DiPrima [BD] for more information about these techniques.

Overall, solutions with initial conditions near the critical point do not stay close to the critical point, so we call this equilibrium unstable, and furthermore we call this specific type (one positive eigenvalue and one negative eigenvalue) of point a saddle point. Two other types of critical points that can arise from real eigenvalues of $A$ are sources and sinks, which are influenced by the eigenvalues of $A$ in the same way saddle points are. Considering a linear system of the form $x' = Ax$, if both the eigenvalues $\lambda_1$ and $\lambda_2$ are positive, then all solutions to the IVP with a nonzero initial condition tend to infinity in magnitude as $t$ tends to infinity, so we call the equilibrium at the origin a source. When both $\lambda_1 < 0$ and $\lambda_2 < 0$, we call the equilibrium at the origin a sink, and in this situation all solutions tend to zero as $t$ tends to infinity because of the exponential decay that is introduced into the expression of the solutions.

Next we discuss the situation where the eigenvalues of $A$ are complex. This will, in turn, lead to two new types of critical points: spirals and centers. We will demonstrate the former with an example here and the latter will be presented in a later section.

Consider the system:

$$x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x. \tag{5}$$

The eigenvalues for this system are $\lambda_1 = -1/2 + i$ and $\lambda_2 = -1/2 - i$. Finding the corresponding eigenvectors $\xi^{(1)}, \xi^{(2)}$ yields two possible solutions of the form $x^{(j)} = \xi^{(j)} e^{\lambda_j t}$ for $j = 1, 2$:

$$x^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} \text{ and } x^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}.$$

However, in this case, the real and imaginary parts of $x^{(2)}$ are linear combinations of the real and imaginary parts of $x^{(1)}$, so we can express the whole solution with just $x^{(1)}$ in the form $\operatorname{Re} x^{(1)} + i \operatorname{Im} x^{(1)}$, or more simply as $x^{(1)} = u(t) + iv(t)$:

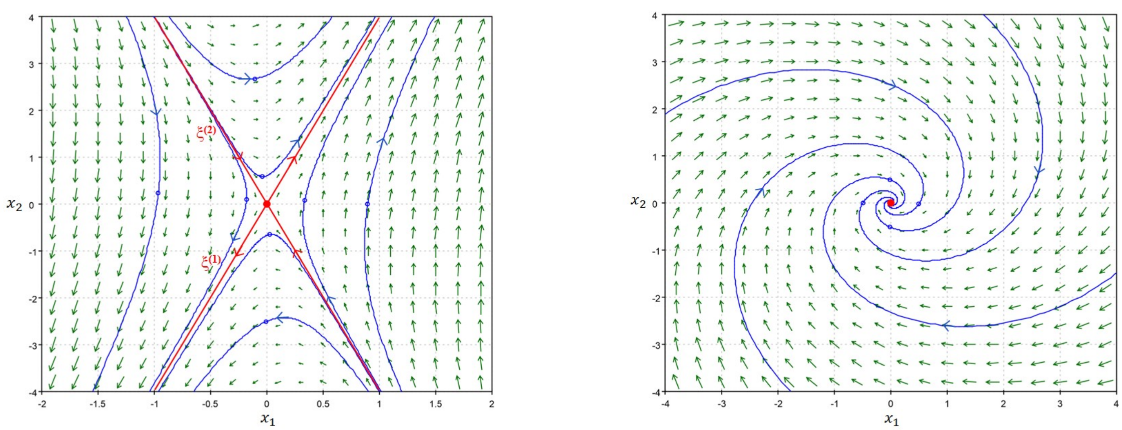
$$x^{(1)} = \underbrace{\begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix}}_{u(t)} + i \underbrace{\begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}}_{v(t)}.$$

As a result of this discussion, any solution to (5) can be written as a linear combination of $u$ and $v$.

While our phase portrait of the system will reveal its spiral nature, it is clear that the solutions differ from the previous straight-line solutions because of the trigonometric factors that are involved. The direction of rotation can also be determined based on $a_{ij}$. In fact, with some elementary computations we can see that the solutions spiral in a clockwise direction when $a_{12} > a_{21}$, and in a counterclockwise direction when $a_{12} < a_{21}$. Thus, for this system, since $a_{12} = 1 > -1 = a_{21}$, the solutions will rotate clockwise about the critical point at the origin.

In Figure 1b below, we have included a phase portrait for the system of equations in (5), along with a few plots of numerical solutions. This phase portrait allows us to see the actual behaviors of solutions to system (5). We should note here that the eigenvalues of $A$ alone

Figure 1




(a) A phase portrait of system (3)

(b) A phase portrait of system (5)

determine the limiting behavior of solutions to the linear system as $t$ tends to infinity. This behavior can be determined independent of the eigenvectors, and we will rely on analyzing the eigenvalues of a linear system when solving explicitly becomes too time consuming or impossible. This occurrence tends to manifest when we are faced with a nonlinear system, and therefore, linearization is another useful tool in the analysis of systems of first-order ODEs that allows us to apply an understanding of eigenvalues to nonlinear systems of ODEs.

## 3.2 Linearization

In the context of this paper, our goal is to begin with a nonlinear system, linearize it at its critical points, then use linear ODE techniques to understand the approximate behavior of solutions to the linearized system, and finally apply our understanding of the behavior to the nonlinear system. As detailed in the previous sections, solving the system $x' = Ax$ is a straightforward process, but the situation is severely complicated if the functions $F_1, F_2$ in (1) are not linear. In this section we discuss a technique known as linearization for systems of ODEs, which is a way to approximate nonlinear systems much in the same way that a tangent line can be used to approximate a smooth function.

Consider again the system of ODEs in (1), where the functions $F_1, F_2$ can be nonlinear in the variables $x_1, x_2$. We also assume that $F_1, F_2$ are continuously differentiable so that our linearization techniques are applicable. Given an equilibrium solution $X$, we linearize the system at the equilibrium by replacing $x$ by $u = x - X$ and replacing $F_j(x)$ by its tangent plane at $X$ for each $j$. Note that $F_j(X) = 0$ since $X$ is an equilibrium solution to the system, and hence the linearized system is given by the following system of ODEs:

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_{x_1} F_1(X) & \partial_{x_2} F_1(X) \\ \partial_{x_1} F_2(X) & \partial_{x_2} F_2(X) \end{pmatrix}}_{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{6}$$

where $u_j(t) = x_j(t) - X_j$ for $j = 1, 2$. Here, we use the partial derivatives of $F_1, F_2$ to linearize the system; that is, the tangent plane of $F_j$ at $X$ is

$$y = F_j(X) + \nabla F_j(X) \cdot (x - X) = \partial_{x_1} F_j(X) u_1 + \partial_{x_2} F_j(X) u_2.$$

Note that $F_j(X) = 0$ here since $X$ is an equilibrium solution. The formula can also be written as the total derivative of the nonlinear system:

$$u' = \underbrace{DF(X)}_{A} u,$$

where $DF(x) = \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)}(x)$ and $X \in \mathbb{R}^2$ is a critical point of the system.

We note here that the method of linearization is not applicable to all nonlinear, first-order systems of ODEs. The matrix of partial derivatives, $A$, must be defined at the equilibria. To guarantee the existence of $A$, $F_1$ and $F_2$ must be differentiable, and they must be continuously differentiable on an open neighborhood of the critical point for much of the existence and uniqueness theory for ODEs to apply. There are also issues that arise with the lack of invertibility, or equivalently vanishing eigenvalues, of $A$ when the critical point is not isolated. This situation presents itself below when we encounter an infinite number of critical points that are not isolated.

# 4 Stability of Linear and Nonlinear Systems

At this point, we have the tools to analyze the equilibria of a system of ODEs based on the coefficient matrix $A$ of either a linear or linearized system. We have seen how different systems can exhibit different types of behaviors at certain critical points, and we have seen that the eigenvalues of $A$ are the determining factors in which type of behavior a critical point will exhibit. Now, we can concisely classify the types of equilibria a system will have as well as their stability based solely on the eigenvalues of the matrix $A$.

## 4.1 Classifying Equilibria

In general, we can predict the behavior of the solutions to a linear or linearized system based on the eigenvalues of the matrix $A$. As discussed earlier, negative eigenvalues cause a solution to tend to zero, positive eigenvalues cause the solution to approach infinity, and imaginary eigenvalues lead to a spiraling behavior of solutions.

For a concise description of the different possibilities for eigenvalues and their corresponding effects on the end behaviors of solutions, see the table below. We omit the case where one eigenvalue is zero and the other less than or equal to zero, however, it is possible to continue the same type of analysis for this situation. We also omit discussion of explicit solutions to $x' = Ax$ when $A$ has repeated eigenvalues. These solutions can be found, but involve ideas of Jordan forms for the matrix $A$. See, for example, the text by Boyce and DiPrima [BD] for more information on these solutions.

| Eigenvalues | Type of Critical Point | Possible end behavior for $x^0 \neq 0$ |
|---|---|---|
| $\lambda_1 \geq \lambda_2 > 0$ | Source | $\|\|x(t)\|\| \to \infty$ |
| $\lambda_1 \leq \lambda_2 < 0$ | Sink | $\|\|x(t)\|\| \to 0$ |
| $\lambda_1 > 0 > \lambda_2$ | Saddle Point | $\|\|x(t)\|\| \to 0$ or $\|\|x(t)\|\| \to \infty$ |
| $\lambda_1, \lambda_2 = r \pm i\mu$ and $r > 0$ | Outward Spiral | $\|\|x(t)\|\| \to \infty$ |
| $\lambda_1, \lambda_2 = r \pm i\mu$ and $r < 0$ | Inward Spiral | $\|\|x(t)\|\| \to 0$ |
| $\lambda_1, \lambda_2 = \pm i\mu$ | Center | $\|\|x(t)\|\| = c$ for some $c \geq 0$ |

For the last line of this table, corresponding to a center, one can conclude that $||x(t)|| = c$ by direct computation since the solution is given by

$$x(t) = c_1 \begin{pmatrix} \cos(\mu t) \\ -\sin(\mu t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\mu t) \\ \cos(\mu t) \end{pmatrix}$$

and

$$||x(t)||^2 = (c_1\cos(\mu t) + c_2\sin(\mu t))^2 + (-c_1\sin(\mu t) + c_2\cos(\mu t))^2 = c_1^2 + c_2^2 = c^2.$$

Because linearization is a good approximation in regions very close to the equilibria, our analysis can also be applied to the nonlinear system in the neighborhood of each critical point. However, when it comes to stability, it is slightly more difficult to directly link the outcomes of the linear analysis to the behavior of the nonlinear system. We are able to make some conclusions about stability of nonlinear equations based on the stability of the associated linearization, but in some situations the linearization is not sufficient to make such conclusions.

## 4.2 Classifying Stability

In the context of this paper, we have seen how the eigenvalues of $A$ influence the behavior of solutions to the linear system, but now we can apply our analysis of the eigenvalues to the stability of the critical points of linear and nonlinear systems. In general, based on our conditions of stability, the types of critical points which attract solutions to them (sinks and inward spirals) are asymptotically stable, the types of critical points which have any sort of repelling nature (sources, saddles, and outward spirals) are unstable, and the remaining type of critical point (centers) only meets the criterion of a stable critical point. Using our criterion of stability, we can adapt the table from the previous section to include stability information about each type of critical point.

Our next task, then, is to broaden our understanding of stability to a nonlinear system. We introduce a little notation first. Given a system of ODEs $x' = F(x)$ with critical point $X \in \mathbb{R}^2$, we say that the system is linearly stable, linearly unstable, or linearly asymptotically stable if the associated linearized system at $X$ is stable, unstable, or asymptotically stable respectively. We will need the next theorem to make stability conclusions about nonlinear systems based on stability analysis of associated linear systems. This theorem can be found for example in the text by Boyce and DiPrima [BD].

**Theorem 2.** *Assume $F(x)$ is a continuously differentiable function on $\mathbb{R}^2$, and that $X$ is an equilibrium solution of $x' = F(x)$. If the linear system $x' = DF(X)x$ is asymptotically stable, then the system is asymptotically stable at $x = X$. If the linear system $x' = DF(X)x$ is unstable, then the system is unstable at $X$.*

Therefore, looking at the eigenvalues of the linear system is a good way to approximate the end behavior of solutions to the nonlinear system in most scenarios. However, because we cannot conclude asymptotic stability, stability, or instability for a critical point whose linearization yields a center using Theorem 2, we cannot conclude anything about its stability in a nonlinear system without going much more in depth. Hence, we can rewrite our eigenvalue table, and here we use the term "linear stability" to describe the stability of critical points in linear or linearized system whereas the term "nonlinear stability" refers to the stabilities of critical points in a nonlinear system.

| Eigenvalues of $DF(X)$ | Type of Critical Point | Linear Stability | Nonlinear Stability |
|---|---|---|---|
| $\lambda_1 \geq \lambda_2 > 0$ | Source | Unstable | Unstable |
| $\lambda_1 \leq \lambda_2 < 0$ | Sink | Asymptotically Stable | Asymptotically Stable |
| $\lambda_1 > 0 > \lambda_2$ | Saddle Point | Unstable | Unstable |
| $\lambda_1, \lambda_2 = r \pm i\mu$, $r > 0$ | Outward Spiral | Unstable | Unstable |
| $\lambda_1, \lambda_2 = r \pm i\mu$, $r < 0$ | Inward Spiral | Asymptotically Stable | Asymptotically Stable |
| $\lambda_1, \lambda_2 = \pm i\mu$ | Center | Stable | Indeterminate |

The classification of equilibria as different types of critical points and as stable or unstable, which is summarized in the table above, is one of the goals of this paper. Now, using linear differential equation theory, we are able to understand a great deal about a system based solely upon the eigenvalues of its coefficient matrix. We will use this sort of analysis in the following phase-portrait characterizations.

# 5 Phase-Portrait Characterizations

Understanding linear differential equation theory makes it possible to, given a complicated nonlinear system, linearize it at its equilibria, classify those equilibria based on the eigenvalues of the system's coefficient matrix $A$, and determine the stability of those equilibria as well as the behaviors of solutions to the system near those equilibria. Therefore, using these techniques, we will analyze two nonlinear models to predict the behaviors of the systems based upon certain parameters. The models we will present can be initially understood in terms of population models in biology. In those models, it is typically only used when solutions that lie in the first quadrant, which correspond to nonnegative values for populations, but we will use our analysis techniques to create phase portraits for these systems on the entire plane. Extending our analysis to all solutions in the plane for these equations is a new contribution to ODE theory, which involves a substantial amount of work resulting in interesting applications of the linearization techniques and new phase plane characterizations for the systems of ODEs introduced later in the this section.

## 5.1 A Simple Nonlinear System of ODEs

In this section we use the techniques detailed in the previous sections to study the following system of ODEs,

$$\begin{aligned} dx/dt &= x(a - \alpha y), \\ dy/dt &= y(-c + \gamma x). \end{aligned} \tag{7}$$

Here we assume that the parameters $a$, $c$, $\alpha$, and $\gamma$ are positive. This system is commonly used as a predator-prey model in biology when the variables $x$ and $y$ are restricted to be nonnegative. In this context, the system (7) is known as the Lotka-Volterra equations. Given an initial condition $(x_0, y_0)$ with nonnegative entries, this system can be interpreted as a population model where $x$ represents the population of the prey, $y$ represents the population of the predator, and the parameters $\alpha$ and $\gamma$ are determined by the birth/death rates and interaction of the two species. More information on this can be found in [BD] and the references therein. For the purposes of our analysis, we simply assume $a, c, \alpha, \gamma > 0$ and consider solutions of the equation that allow $x$ and $y$ to be arbitrary real-valued solutions, with initial conditions $(x_0, y_0)$ lying anywhere in the plane $\mathbb{R}^2$. All of the results discussed in this section pertaining to nonnegative solutions to (7), as well as a more detailed explanation of the physical meanings of the parameters can be found in the text by Boyce and DiPrima [BD].

To analyze system in (7) it helps to approach it in more general terms first, and then to look at specific situations. Looking at system (7), one can see that there are critical points at $(0, 0)$ and at $(c/\gamma, a/\alpha)$, and applying the linearization in (6) at those points one can observe the approximate behavior of the solutions over time when an initial condition is close to $(0, 0)$ or $(c/\gamma, a/\alpha)$. We start by computing the derivative matrix

$$DF(x, y) = \begin{pmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix}. \tag{8}$$

The linearization at the equilibrium solution $(0, 0)$ is given by $DF(0, 0)$,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The eigenvalues for this system, then, are $\lambda_1 = a$ and $\lambda_2 = -c$. Because $a$ and $c$ are both positive, it becomes clear that $(0, 0)$ is a saddle point (since one eigenvalue is positive and the other is negative). Hence $(0, 0)$ is an unstable equilibrium of the predator-prey system.

At the point $(c/\gamma, a/\alpha)$, the system can be analyzed in a similar fashion. The linearized system is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & \frac{-c\alpha}{\gamma} \\ \frac{a\gamma}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

for which the eigenvalues are $\lambda_1 = \sqrt{ca}\, i$ and $\lambda_2 = -\sqrt{ca}\, i$. Looking at our stability table, we see that this critical point is a center; hence the equilibrium $(c/\gamma, a/\alpha)$ is linearly stable, but we are unable to conclude the stability for the general nonlinear system at this equilibrium.

Our next step, then, is to look at the shape of the center in the phase plane to get a better picture of how the levels of the two populations relate to one another. The shape of the solution near $(c/\gamma, a/\alpha)$ can be determined through observing the relationship $dv/du$. Starting with the relation

$$\frac{dv}{du} = -\frac{a\gamma^2 u}{c\alpha^2 v}, \tag{9}$$

we can simplify and integrate to obtain the equation

$$\frac{u^2}{K/(a\gamma^2)} + \frac{v^2}{K/(c\alpha^2)} = 1,$$

where $K$ is a constant of integration.

One can easily recognize the form of this equation as an ellipse, and it makes sense that this form would appear as it corresponds to the general shape of the center. Looking at the equation for the ellipse, now, we can assess the placement of the major and minor axes based on the values for $a\gamma^2$ and $c\alpha^2$. That is, if $a\gamma^2 > c\alpha^2$, then we will observe a vertical stretch in the elliptical shape of solutions around the point $(c/\gamma, a/\alpha)$, and if $a\gamma^2 < c\alpha^2$, then we will observe a horizontal stretch. If $a\gamma^2 = c\alpha^2$, then the solutions will take the shapes of circles about the point $(c/\gamma, a/\alpha)$.

Seeing that $(c/\gamma, a/\alpha)$ is a center, we can find which way the solutions of the two-dimensional system will move about the critical point. Based on the analysis done for system (5), for the point $(c/\gamma, a/\alpha)$ if $\frac{c\alpha}{\gamma} - \frac{a\gamma}{\alpha} > 0$, then the solutions will move clockwise about the critical point, and if $\frac{c\alpha}{\gamma} - \frac{a\gamma}{\alpha} < 0$, then the solutions will move counterclockwise about the critical point.

One final aspect of the behaviors of solutions that we can identify qualitatively is the eccentricity of the solutions. That is, the ratio of the lengths of the major axes to those of the minor axes. If we express the length of the horizontal axis as $z$ and the length of the vertical axis as $w$, then we can define the eccentricity of the ellipse as

$$\frac{z}{w} = \sqrt{\frac{c\alpha^2}{a\gamma^2}}.$$

Through this qualitative analysis we have now determined that $(0, 0)$ is an unstable equilibrium of the linearized system and that $(c/\gamma, a/\alpha)$ is a stable, but not asymptotically stable, equilibrium of the linearized system that exhibits a center-like behavior. Now given particular values for the constants of the system, we can immediately predict the long-term behavior of the two populations. To observe the results of our analysis, consider the system:

$$\begin{aligned} dx/dt &= x(1.5 - 0.5y) \\ dy/dt &= y(-0.5 + x) \end{aligned} \tag{10}$$

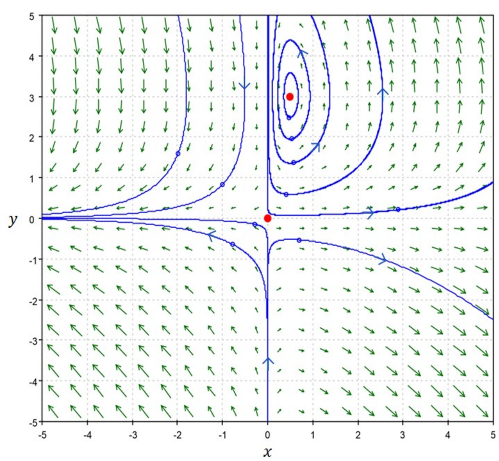
We can immediately observe that $a = 1.5$, $\alpha = 0.5$, $c = 0.5$, and $\gamma = 1$. The critical points, then, are $(0, 0)$ and $(c/\gamma, a/\alpha) = (0.5, 3)$. The former critical point will be a saddle. The latter critical point will be a center, and if we do some minor calculations we can see that $a\gamma^2 = 1.5$ and $c\alpha^2 = 0.125$, so the solutions will tend to be vertically stretched. If we want to know how vertically stretched the solutions are, we can look at the ratio $z/w$, which in this case can be computed as follows:

$$\frac{z}{w} = \sqrt{\frac{0.125}{1.5}} = \frac{1}{\sqrt{12}}.$$

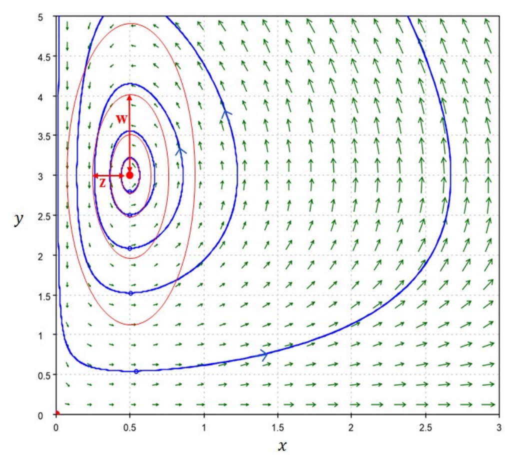
Therefore, the vertical axes of the elliptical solutions will be longer than the corresponding horizontal axes by a factor of $\sqrt{12} \approx 3.46$. Thus, the major axes can be expressed by $w$ and the minor axes can be expressed by $z$, and we should note here that as we look at solutions that are further from the critical point, the ratio $z/w$ will become a less accurate representation of a relationship between the two lengths because we are working with a nonlinear system that we have approximated with our linearization techniques. We can also observe that $\frac{c\alpha}{\gamma} - \frac{a\gamma}{\alpha} = 0.75 - 1 = -0.25 < 0$, so the vertically-stretched elliptical solutions will move counterclockwise about the critical point.

A phase portrait can be used to verify our analysis, and by looking at Figures 2a and 2b, we can see that the critical points, the direction and magnitude of stretching at the center point, as well as the direction of the solutions all match our predictions. Here, the figure on the left is a phase plane of the system that reveals the effects of the equilibria on solutions in all quadrants of the $x$-$y$ plane. The figure on the right shows only the first quadrant since historically the purpose of this model is to describe positive values for the two populations. In Figure 2b, we also present the elliptical solutions of the linearized system which (not so surprisingly) become more accurate representations of solutions to the nonlinear system in regions closer to the equilibrium $(c/\gamma, a/\alpha)$.

Figure 2



(a) A phase portrait of system (10)



(b) First quadrant including linearized solutions

After some analysis, it is possible to determine the behaviors of solutions to the system near the equilibria with very few calculations. It is difficult, though, to make claims about the long-term behaviors of solutions since our analysis relies on approximations that are only valid in the neighborhoods of the equilibria. While we cannot necessarily extrapolate the long-term behavior from our analysis, we can still know about the locations and types of equilibria, as well as the general shapes and directions of solutions near those equilibria. This predator-prey model is one example of the power of being able to properly analyze linear systems because the simple techniques we apply to the linear system allow us to know a great deal about the behaviors of solutions to the nonlinear system.

As a closing remark, it it important to note that in the case of a center in the linear system, the stability of the nonlinear system cannot be directly determined from the techniques discussed in this paper. We can easily predict the behavior of solutions in the neighborhood of critical points, but not much can be said about the stability of centers in nonlinear systems without going more in-depth.

### 5.2 A More Complicated Nonlinear System of ODEs

The second system of ODEs we consider is

$$\begin{aligned} dx/dt &= x(\alpha_1 - \beta_1 x - \gamma_1 y), \\ dy/dt &= y(\alpha_2 - \beta_2 x - \gamma_2 y), \end{aligned} \tag{11}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$. This model again is founded in biological population modeling. When one restricts to nonnegative solutions $x, y \geq 0$, this system can be interpreted as a competing-species model where $x$ and $y$ represent the populations of two species competing for a common food source. In this setting, the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ would be determined by environmental factors and characteristics of the particular species being modeled. More information on this can be found in the text by Boyce and DiPrima [BD] and the references therein. We will consider solutions to the equation in the whole plane $\mathbb{R}^2$, rather than restricting to the first quadrant, which yields some interesting new results in the behavior of solutions to the system. The results below can again be found in the text by Boyce and DiPrima [BD] when solutions are restricted to being nonnegative. Although, here we formulate a full analysis of (11) for arbitrary solutions (not just nonnegative solutions), which as we will see, poses a more formidable challenge.

We start our analysis by making a change of variables with $u = \beta_1 x$ and $v = \gamma_2 y$. Thus, we arrive at the system:

$$\begin{aligned} u' &= u(\alpha_1 - u - (\gamma_1/\gamma_2)v), \\ v' &= v(\alpha_2 - (\beta_2/\beta_1)u - v), \end{aligned} \tag{12}$$

If we let $\gamma = \gamma_1/\gamma_2$ and $\beta = \beta_2/\beta_1$, the system simplifies to a point where we can more easily apply the linearization techniques we have been discussing. We note that the solutions to equation (11) and (12) are identical up to a rescaling of the variables, in particular using the substitution $u = \beta_1 x$ and $v = \gamma_2 y$. So from this point on, we only consider equation (12), with the understanding that solutions to (11) are completely described by those of the simplified ones in (12). The simplified system in (12) written in a more concise notation becomes

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = F(u, v) = \begin{pmatrix} u(\alpha_1 - u - \gamma v) \\ v(\alpha_2 - \beta u - v) \end{pmatrix},$$

and we can find the equilibria by looking at where $u' = v' = 0$.

Observe that $u' = 0$ where $u = 0$ and where $u = \alpha_1 - \gamma v$, and that $v' = 0$ where $v = 0$ and where $v = \alpha_2 - \beta u$. These equations are lines in the $u$-$v$ plane, and wherever a line on which $u' = 0$ crosses a line on which $v' = 0$, the system will have an equilibrium. For simplicity, we will be referring to $u = 0$ as $L_1$, $u = \alpha_1 - \gamma v$ as $L_2$, $v = 0$ as $L_3$, and $v = \alpha_2 - \beta u$ as $L_4$.

The system will have a certain number of equilibria that is dependent upon how the given values for the parameters $\alpha_1, \alpha_2, \beta$, and $\gamma$ cause $L_2$ and $L_4$ to be oriented in the plane. Because we are looking at two lines in a plane, there are only three possibilities: $L_2$ and $L_4$ could be parallel, they could be one and the same line, or they could intersect at one point.

If $L_2$ and $L_4$ are parallel, then the system only has three equilibria: where $L_2$ intersects $L_3$ at $(\alpha_1, 0)$, where $L_1$ intersects $L_4$ at $(0, \alpha_2)$, and where $L_1$ intersects $L_3$ at the origin. For $L_2$ and $L_4$ to be parallel (but not coincide), it must be the case that $\beta\gamma = 1$ and $\beta \neq \alpha_1/\alpha_2$. Under these conditions, the $u$-$v$ plane is pictured in Figure 3a below.

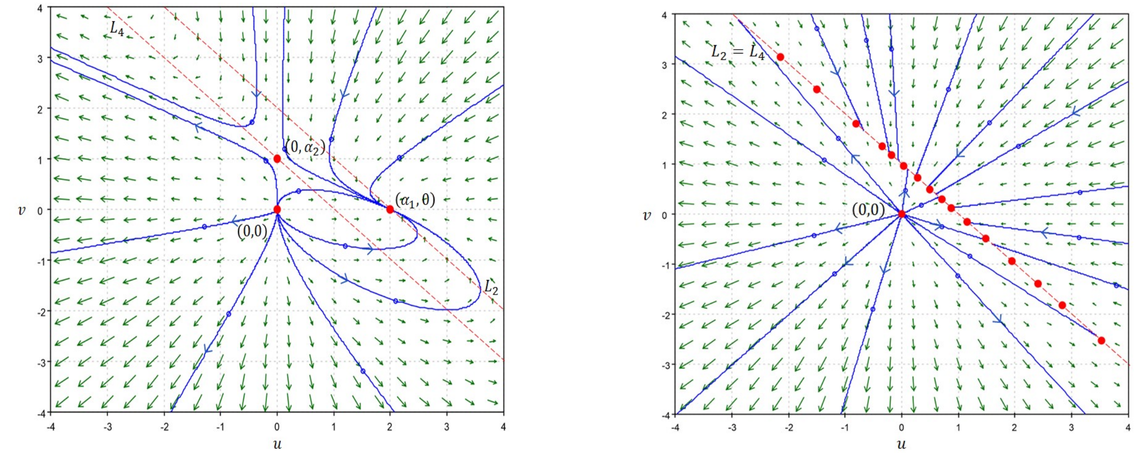
By picking initial conditions in each region of the plane we can gain a qualitative understanding how solutions to the system will behave, and the behaviors of the system are illustrated by the arrows. With this type of analysis is it possible to see that near the origin all solutions will be repelled, near $(0, \alpha_2)$ solutions will be attracted in one direction and repelled in the other, and near $(\alpha_1, 0)$ all solutions will be attracted. Notice how there is no equilibrium located where both of the populations have positive numbers of members, so we know that two competing species could not coexist if $\beta\gamma = 1$ and $\beta \neq \alpha_1/\alpha_2$, according to this model. However, in a broader scope we can appreciate that $(0, 0)$ will be a source, that $(0, \alpha_2)$ will be a saddle, and that $(\alpha_1, 0)$ will be a sink.

The second possibility for $L_2$ and $L_4$ in the $u$-$v$ plane is that they are one and the same line. For this to happen, $\beta\gamma = 1$ and $\beta = \alpha_1/\alpha_2$. When $L_2$ and $L_4$ coincide, there would exist one equilibrium at the origin and infinitely many equilibria along $L_2$ (or $L_4$), which is to say that the equilibria exist at $(0, 0)$ and at $(t, \alpha_2 - \beta t)$ for all $t \in \mathbb{R}$. The phase portrait for the $u$-$v$ plane in this case is given by Figure 3b.

The numerical solution in Figure 3b indicates that solutions to the system are drawn toward $L_2$ (or $L_4$), however, this conclusion cannot be reached using the linearization techniques of this paper. It is not hard to see that since the equilibria along $L_2$ are not isolated, they cannot be asymptotically stable. Hence it is clear that Theorem 2 is not well-suited to analyze the stability in this situation. In fact, the numerical simulations indicate that these equilibria are stable (though we cannot make this conclusion rigorously). So linearization techniques do not appear to be appropriate for analysis of the ODE when $L_2 = L_4$. In our context, though, we can at least make the observation that $(0, 0)$ will be a source and that to gain a better understanding the behaviors of solutions near $L_2$ (or $L_4$) we will have to turn to other methods.

The final possibility would be when $L_2$ and $L_4$ intersect at only one point, which occurs when $\beta\gamma \neq 1$. In this scenario, there are a total of four equilibria at the points $P_1(0, 0)$, $P_2(\alpha_1, 0)$, $P_3(0, \alpha_2)$, and $P_4(\frac{\alpha_1 - \gamma\alpha_2}{1 - \beta\gamma}, \frac{\alpha_2 - \beta\alpha_1}{1 - \beta\gamma})$. The last equilibrium is of particular interest because it is the only equilibrium to have a chance of existing at a location where both of the populations would have a nonzero number of members. In saying that, the fourth equilibrium could exist in the first, second, or fourth quadrants of the $u$-$v$ plane, because both $L_2$ and $L_4$ have negative slopes and are positive-valued when intersecting the $v$-axis. Though, no matter the quadrant in which $P_4$ is located, we can determine the type of critical point by examining the parameters of the system as we did with the predator-prey model. Thus, we will linearize our model and apply the techniques of this paper. To linearize the

Figure 3




(a) Simulation of the $u$-$v$ plane when $L_2 \parallel L_4$

(b) Simulation of the $u$-$v$ plane for $L_2 = L_4$

system we take the total derivative of $F(u,v)$:

$$DF(u,v) = \begin{pmatrix} \alpha_1 - 2u - \gamma v & -\gamma u \\ -\beta v & \alpha_2 - \beta u - 2v \end{pmatrix}. \tag{13}$$

Thus, at $P_1$ the linearized system is:

$$w' = DF(0,0)w = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} w, \text{ where } w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

The eigenvalues for $DF(0,0)$ are $\lambda_1 = \alpha_1$ and $\lambda_2 = \alpha_2$, which must both be positive. Therefore, the critical point will be unstable because it will repel all nearby solutions.

We should note here that for our future analysis we will let $c_1 = \alpha_1 - \gamma\alpha_2$ and $c_2 = \alpha_2 - \beta\alpha_1$. At $P_2$ the linearized system is:

$$w' = DF(\alpha_1,0)w = \begin{pmatrix} -\alpha_1 & -\gamma\alpha_1 \\ 0 & \alpha_2 - \beta\alpha_1 \end{pmatrix} w = \begin{pmatrix} -\alpha_1 & -\gamma\alpha_1 \\ 0 & c_2 \end{pmatrix} w.$$

The eigenvalues for $DF(\alpha_1,0)$ are $\lambda_1 = -\alpha_1$ and $\lambda_2 = c_2$. In this case, if $c_2 < 0$, then $\lambda_1, \lambda_2 < 0$ and the critical point is asymptotically stable. However, if $c_2 > 0$, then $\lambda_1 < 0 < \lambda_2$, so the critical point will be a saddle point and hence be unstable.

At $P_3$ the system is:

$$w' = DF(0,\alpha_2)w = \begin{pmatrix} \alpha_1 - \gamma\alpha_2 & 0 \\ -\beta\alpha_2 & -\alpha_2 \end{pmatrix} w = \begin{pmatrix} c_1 & 0 \\ -\beta\alpha_2 & -\alpha_2 \end{pmatrix} w.$$

The eigenvalues for $DF(0,\alpha_2)$ are $\lambda_1 = c_1$ and $\lambda_2 = -\alpha_2$. Therefore, if $c_1 < 0$, then the critical point will be linearly stable because $\lambda_1, \lambda_2 < 0$. If $c_1 > 0$, however, then the critical point will be unstable because $\lambda_2 < 0 < \lambda_1$.

From our analysis of the first three equilibria and how they are affected by the signs of $c_1$ and $c_2$, we can learn about the stability of the equilibria by finding to which quadrant of the $c_1$-$c_2$ plane the parameters of the system correspond. The results of this analysis can be organized into the following table.

| Location of $(c_1, c_2)$ | $P_1$ | $P_2$ | $P_3$ |
|---|---|---|---|
| $c_1, c_2 > 0$ | Source | Saddle | Saddle |
| $c_1 < 0 < c_2$ | Source | Saddle | Sink |
| $c_1, c_2 < 0$ | Source | Sink | Sink |
| $c_1 > 0 > c_2$ | Source | Sink | Saddle |

Table 1: Stability of $P_1$, $P_2$, and $P_3$ based on location in the $c_1$-$c_2$ plane.

We can look at $P_4$ in the same fashion as $P_1$, $P_2$, and $P_3$, however, because the system will have more complicated eigenvalues, the stability analysis will be a little more extensive. For $P_4$ the linearized system is:

$$w' = DF\left(\frac{c_1}{1-\beta\gamma}, \frac{c_2}{1-\beta\gamma}\right) w = \frac{1}{1-\beta\gamma}\begin{pmatrix} \gamma\alpha_2 - \alpha_1 & \gamma(\gamma\alpha_2 - \alpha_1) \\ \beta(\beta\alpha_1 - \alpha_2) & \beta\alpha_1 - \alpha_2 \end{pmatrix} w$$
$$= \frac{1}{\beta\gamma - 1}\begin{pmatrix} c_1 & \gamma c_1 \\ \beta c_2 & c_2 \end{pmatrix} w,$$

where again $c_1 = \alpha_1 - \gamma\alpha_2$ and $c_2 = \alpha_2 - \beta\alpha_1$. The eigenvalues of $DF\left(\frac{c_1}{1-\beta\gamma}, \frac{c_2}{1-\beta\gamma}\right)$ can be found by using the quadratic formula to solve for the roots of the characteristic polynomial of the coefficient matrix as follows: The characteristic polynomial of $DF(\frac{c_1}{1-\beta\gamma}, \frac{c_2}{1-\beta\gamma})$ is

$$\left(\lambda - \frac{c_1}{\beta\gamma - 1}\right)\left(\lambda - \frac{c_2}{\beta\gamma - 1}\right) - \frac{\beta\gamma c_1 c_2}{(\beta\gamma - 1)^2} = \lambda^2 - \frac{c_1 + c_2}{\beta\gamma - 1}\lambda - \frac{c_1 c_2}{\beta\gamma - 1},$$

and the roots of the characteristic polynomial are

$$\lambda_1, \lambda_2 = \frac{(c_1 + c_2) \pm \sqrt{(c_1 + c_2)^2 + 4(\beta\gamma - 1)c_1 c_2}}{2(\beta\gamma - 1)}.$$

To determine which values of each constant will cause the critical point to be a source, sink, saddle, etc., we need to look at the eigenvalues $\lambda_1, \lambda_2$ more closely. The first distinction we make is where the eigenvalues are real and where they are complex. To accomplish this task, we let

$$g(c_1, c_2) = (c_1 + c_2)^2 + 4(\beta\gamma - 1)c_1 c_2 = c_1^2 + c_2^2 + 2(2\beta\gamma - 1)c_1 c_2,$$

which is the discriminant of the quadratic equation above, and assess where $g(c_1, c_2) \geq 0$ and where $g(c_1, c_2) < 0$. If $\beta\gamma < 1$, it follows that $g(c_1, c_2) \geq 0$ for all $c_1, c_2 \in \mathbb{R}$. If $\beta\gamma > 1$, then it is possible that $g(c_1, c_2)$ is negative. If $\beta\gamma = 1$, then we fall into one of the two situations above, where either $L_2$ and $L_4$ are parallel or equal (see Figures 3a and 3b). So we assume that $\beta\gamma \neq 1$, and split into two situations, where $\beta\gamma < 1$ and $\beta\gamma > 1$.

For future reference, we also define

$$f_{\pm}(c_1, c_2) = \frac{c_1 + c_2 \pm \sqrt{g(c_1, c_2)}}{2(\beta\gamma - 1)},$$

which gives the eigenvalues $\lambda_1, \lambda_2 = f_{\pm}(c_1, c_2)$.

### 5.2.1 The Case $\beta\gamma < 1$

We first note that by our assumptions, it follows that $\alpha_1, \alpha_2, \beta, \gamma > 0$. So in the case that $\beta\gamma < 1$, it cannot be the case that both $c_1 < 0$ and $c_2 < 0$. This is because when $\beta\gamma < 1$, it follows that $c_1 = \alpha_1 - \gamma\alpha_2 < 0$ implies that $\alpha_1 < \gamma\alpha_2$ and hence

$$c_2 = \alpha_2 - \beta\alpha_1 > \alpha_2 - \beta\gamma\alpha_2 = (1 - \beta\gamma)\alpha_2 > 0.$$

Similarly $c_2 < 0$ implies $c_1 > 0$. Therefore if $\beta\gamma < 1$, the only possible values for $(c_1, c_2)$ lie in the first, second, and fourth quadrants. This can be seen geometrically as the fact that the order pair $(\frac{c_1}{1-\beta\gamma}, \frac{c_2}{1-\beta\gamma})$ is the intersection of lines $L_2$ and $L_4$ from the previous section. Since $L_2$ and $L_4$ cannot intersect in the third quadrant, it is impossible for both $c_1$ and $c_2$ to be negative when $\beta\gamma < 1$.

When $\beta\gamma < 1$, it follows that $g(c_1, c_2) \geq 0$ for all $c_1, c_2 \in \mathbb{R}$. Therefore $\lambda_1$ and $\lambda_2$ are always real. It is also not hard to check that $f_{\pm}(c_1, c_2) = 0$ exactly when $c_1 = 0$ or $c_2 = 0$. Therefore we consider the sign of $f_+(c_1, c_2)$ and $f_-(c_1, c_2)$ in each of the three viable quadrants of the $c_1$-$c_2$ plane (the first, second, and fourth quadrants). We already know that $g(c_1, c_2)$ is nonnegative in this situation. Note that $\sqrt{g(c_1, c_2)} < |c_1 + c_2|$ when $c_1$ and $c_2$ have the same sign (that is in the first quadrant). Therefore $\lambda_1 = f_+(c_1, c_2) < 0$ and $\lambda_2 = f_-(c_1, c_2) < 0$ when $(c_1, c_2)$ is in the first quadrant. Also if $(c_1, c_2)$ is in either the second or fourth quadrants, then $\sqrt{g(c_1, c_2)} > |c_1 + c_2|$. Then for $(c_1, c_2)$ in either the second or fourth quadrants, we have $\lambda_2 = f_-(c_1, c_2) < 0 < f_+(c_1, c_2) = \lambda_1$. Thus, we have the stability for $P_4$ in Table 2 for the case $\beta\gamma < 1$. That is, Table 2 characterizes the stability of equilibrium $P_4$ when $\beta\gamma < 1$ based on the location of $(c_1, c_2)$ in the $c_1$-$c_2$ plane.

| Location of $(c_1, c_2)$ | Sign of eigenvalues | Classification of $P_4$ | Stability when $\beta\gamma < 1$ |
|---|---|---|---|
| $c_1, c_2 > 0$ | $\lambda_1, \lambda_2 < 0$ | Sink | Stable |
| $c_1 < 0 < c_2$ | $\lambda_2 < 0 < \lambda_1$ | Saddle | Unstable |
| $c_1 > 0 > c_2$ | $\lambda_2 < 0 < \lambda_1$ | Saddle | Unstable |

Table 2: Stability of $P_4$ when $\beta\gamma < 1$. Note that the situation $c_1, c_2 < 0$ is not possible.

### 5.2.2 The Case $\beta\gamma > 1$

Similar to the last case where $\beta\gamma < 1$, we can immediately conclude some additional information of about $c_1$ and $c_2$ based only on the information $\alpha_1, \alpha_2, \beta, \gamma > 0$ and $\beta\gamma > 1$. In this situation, it is impossible for both $c_1$ and $c_2$ be positive. Again, this can be interpreted geometrically by the following argument. Since $(\frac{c_1}{1-\beta\gamma}, \frac{c_2}{1-\beta\gamma})$ is the intersection of line $L_2$ and $L_4$, it cannot lie in the third quadrant. Since in this case $1 - \beta\gamma < 0$, it is not possible for both $c_1$ and $c_2$ to be positive. Hence we immediately rule out this option for consideration in the stability analysis of equilibrium $P_4$.

Our process here is to determine which regions in the $c_1$-$c_2$ plane correspond to each type of critical point. One can easily check that $g(c_1, c_2)$ can be factored in the following way when $\beta\gamma > 1$:

$$g(c_1, c_2) = \left(c_2 + \left(2\beta\gamma - 1 + 2\sqrt{\beta\gamma(\beta\gamma - 1)}\right) c_1\right)\left(c_2 + \left(2\beta\gamma - 1 - 2\sqrt{\beta\gamma(\beta\gamma - 1)}\right) c_1\right).$$

Therefore, if we solve the equation $g(c_1, c_2) = 0$, we find that the $c_1$-$c_2$ plane becomes divided by the lines:

$$c_2 = mc_1 \text{ and } c_2 = m^{-1}c_1, \text{ where } m = 2\beta\gamma - 1 + 2\sqrt{\beta\gamma(\beta\gamma - 1)}. \tag{14}$$

Since $g(c_1, c_2) = 0$ for any $c_2 = mc_1$, it follows by symmetry that $g(c_1, c_2) = 0$ for any $c_1 = mc_2$. In fact, it it is not hard to show directly that $m^{-1} = 2\beta\gamma - 1 - 2\sqrt{\beta\gamma(\beta\gamma - 1)}$; that is, $g$ can be factored $g(c_1, c_2) = (c_2 - mc_1)(c_2 - m^{-1}c_1)$. This factorization makes it possible to quickly determine the sign of $g(c_1, c_2)$ for any values of $c_1, c_2$. In fact, it splits the $c_1$-$c_2$ plane into 4 regions, where $c_1 - mc_1$ and $mc_2 - c_1$ are positive or negative.

By looking more closely at $f_\pm(c_1, c_2) = \frac{(c_1+c_2)\pm\sqrt{g(c_1,c_2)}}{2(\beta\gamma-1)}$, we can identify where the boundary conditions for each type of critical point are. We find that when $g(c_1, c_2) < 0$, if $c_1 + c_2 < 0$ then we have an inward spiral, if $c_1 + c_2 > 0$ then we have an outward spiral, and if $c_1 + c_2 = 0$ then we have a center. Using this knowledge we can divide the $c_1$-$c_2$ plane even further. We note here that region $I$ is not plausible since $c_1$ and $c_2$ cannot both be positive when $\beta\gamma > 1$.

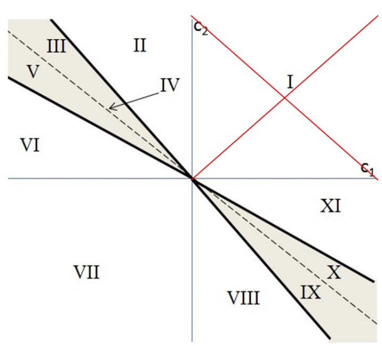



Figure 4: The $c_1$-$c_2$ plane divided into eleven (ten possible) regions for $\beta\gamma > 1$.

If we solve the equation $f_\pm(c_1, c_2) = 0$, we find that $f$ is zero along the $c_1$-and-$c_2$ axes, meaning that $f_\pm$ could potentially change sign when crossing those boundaries. These new boundaries in addition to the boundaries in the imaginary regions will divide the $c_1$-$c_2$ plane into eleven regions for us to analyze. By looking at the sign of the real part of $f_\pm$ (which coincide with looking at the real part of the eigenvalues $\lambda_1, \lambda_2$) in each of these regions, we can tell what type of critical point the system will have when its parameters characterize it as belonging to a certain region of the $c_1$-$c_2$ plane. In total, we arrive at the decomposition of the $c_1$-$c_2$ plane in Figure 4, which we use to determine the sign of $\mathrm{Re}(f_\pm)$ and hence the stability of $P_4$ when $\beta\gamma > 1$.

At this point we can organize our results for $\beta\gamma > 1$ into two tables. To utilize the conditions of these tables, recall that we set $c_1 = \alpha_1 - \gamma\alpha_2$, $c_2 = \alpha_2 - \beta\alpha_1$, $\beta = \beta_2/\beta_1$, $\gamma = \gamma_1/\gamma_2$, $m = 2\beta\gamma - 1 + 2\sqrt{\beta\gamma(\beta\gamma - 1)}$, and $m^{-1} = 2\beta\gamma - 1 - 2\sqrt{\beta\gamma(\beta\gamma - 1)}$. The first step should be to calculate the quadrant of the $c_1$-$c_2$ plane to which the parameters of the system refer. Next, compute the values of $m$, $m^{-1}$, and $|c_2|/|c_1|$, and compare them using Table 3 to determine the correct region out of the possible ten. Once the region has been determined, use Table 4 to determine the behavior of the solutions to the system near the fourth equilibrium. Hence using Tables 3 and 4 we can determine the stability of equilibrium $P_4$ based on a few simple computations involving the parameters in (12).

| | $c_1 < 0 < c_2$ | $c_2 < 0 < c_1$ | $c_1, c_2 < 0$ |
|---|---|---|---|
| $\frac{\lvert c_2\rvert}{\lvert c_1\rvert} > m$ | II | VIII | VII |
| $m > \frac{\lvert c_2\rvert}{\lvert c_1\rvert} > 1$ | III | IX | VII |
| $\frac{\lvert c_2\rvert}{\lvert c_1\rvert} = 1$ | IV | IV | VII |
| $1 > \frac{\lvert c_2\rvert}{\lvert c_1\rvert} > m^{-1}$ | V | X | VII |
| $m^{-1} > \frac{\lvert c_2\rvert}{\lvert c_1\rvert}$ | VI | XI | VII |

Table 3: Description of ten plausible regions of the $c_1$-$c_2$ plane when $\beta\gamma > 1$. Note that it is not possible for $c_1, c_2 > 0$.

### 5.2.3 Numerical Simulations for Equation 11 when $\beta\gamma < 1$

We can now explore phase portraits for the system in equation (11) based on the parameters $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\gamma_1$, and $\gamma_2$. Recall that we've defined $u = \beta_1 x$, $v = \gamma_2 y$, $\gamma = \gamma_1/\gamma_2$, $\beta = \beta_2/\beta_1$, $c_1 = \alpha_1 - \gamma\alpha_2$, and $c_2 = \alpha_2 - \beta\alpha_1$.

We consider the numerical example of equation (11) given by

$$\begin{aligned} dx/dt &= x(1.5 - x - 0.5y), \\ dy/dt &= y(2 - 0.75x - y). \end{aligned} \tag{15}$$

Right away we can see that $\alpha_1 = 1.5$, $\alpha_2 = 2$, $\beta_1 = 1$, $\beta_2 = 0.75$, $\gamma_1 = 0.5$, and $\gamma_2 = 1$. Thus, $c_1 = 0.5$, $c_2 = 0.875$, and $\beta\gamma = 0.375 < 1$. We should also note here that, based on these

| Region | Critical Point | Nonlinear Stability |
|---|---|---|
| $II$ | Source | Unstable |
| $III$ | Outward Spiral | Unstable |
| $IV$ | Center | Indeterminate |
| $V$ | Inward Spiral | Asymptotically Stable |
| $VI$ | Sink | Asymptotically Stable |
| $VII$ | Saddle | Unstable |
| $VIII$ | Sink | Asymptotically Stable |
| $IX$ | Inward Spiral | Asymptotically Stable |
| $X$ | Outward Spiral | Unstable |
| $XI$ | Source | Unstable |

Table 4: Stability of $P_4$ when $\beta\gamma > 1$ depending on location in the $c_1$-$c_2$ plane.

given values of parameters, when we shift to $u$-$v$ coordinates with the substitutions $u = \beta_1 x$, $v = \gamma_2 y$, $\gamma = \gamma_1/\gamma_2$, and $\beta = \beta_2/\beta_1$, the system would coincidentally take the form

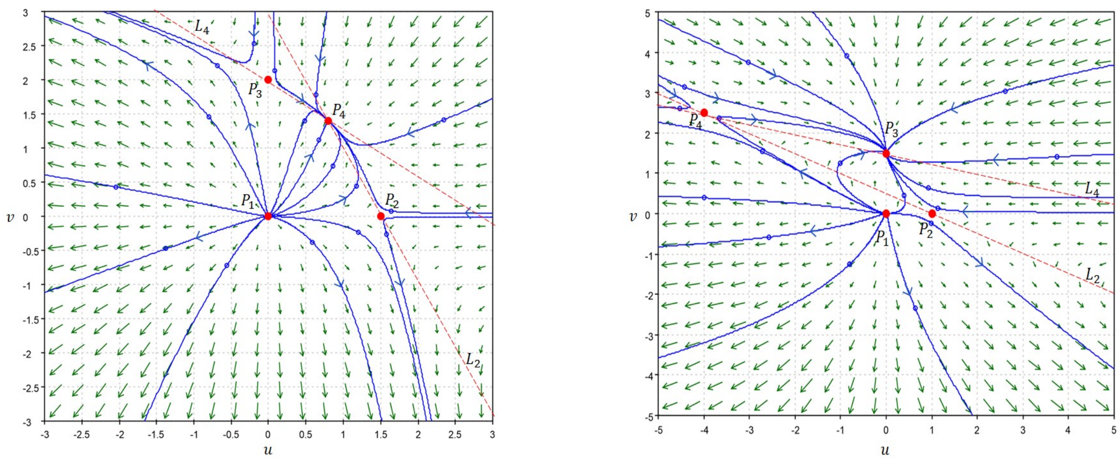
$$du/dt = u(1.5 - u - 0.5v),$$
$$dv/dt = v(2 - 0.75u - v).$$

Thus, we can directly apply out analysis to system (15).

Plugging the former parameters into our equations for the critical points we get the points $P_1(0,0)$, $P_2(1.5,0)$, $P_3(0,2)$, and $P_4(0.5,1.5)$. Now looking specifically at each critical point, and the fact that in this case $\beta\gamma < 1$, we can tell from Table 1 that $P_1$ is a source and $P_2$ and $P_3$ are both saddles. We know that $(c_1, c_2)$ lies in the first quadrant of the $c_1$-$c_2$ plane for this example, that is $c_1, c_2 > 0$, and since $\beta\gamma < 1$ as well, it follows that $P_4$ also lies in the first quadrant. Furthermore, looking at Table 2, it is now easy to see that equilibrium $P_4$ is a sink. Once we have gone through this sort of extensive analysis, it becomes very straightforward to characterize a system based on its parameters, and the techniques used in this section can be applied to other equally-complicated models as well. We can verify our analysis with a phase portrait of the nonlinear system. In Figure 5a, we can see that the equilibria not only exist where we predicted them to be, but also that they are the types of equilibria that we expected.

Using the above analysis of the fourth equilibrium, $P_4$, we can look more closely at the behavior of the system specifically when $(c_1, c_2)$ lies in the second quadrant of the $c_1$-$c_2$ plane, that is, $c_1 < 0 < c_2$. Since $c_1 < 0 < c_2$ and $\beta\gamma < 1$, we know that $\frac{c_1}{1-\beta\gamma} < 0$ and $\frac{c_2}{1-\beta\gamma} > 0$. Therefore $P_4$ lies in the second quadrant of the $u$-$v$ plane. It can also easily be read from Tables 1 and 2, using $c_1 < 0 < c_2$ and $\beta\gamma < 1$, that $P_1$ is a source, $P_2$ is a saddle, $P_3$ is a sink, and $P_4$ is a saddle. This situation is represented in Figure 5b. This analysis can be done for $(c_1, c_2)$ in any viable quadrant for $(c_1, c_2)$–that is in the first, second, or fourth quadrants–to analyze the solutions of (15), and more generally solutions of (11) and (12) when $\beta\gamma < 1$.

Figure 5



(a) Phase portrait and numerical solutions for equation (11) when $c_1, c_2 > 0$ and $\beta\gamma < 1$.

(b) Phase portrait and numerical solutions for equation (11) when $c_1 < 0 < c_2$ and $\beta\gamma < 1$.

#### 5.2.4 Numerical Simulations for Equation 11 when $\beta\gamma > 1$

In this section, we assume that we are in the situation where $\beta\gamma > 1$. So we can use Tables 1, 3, and 4 to determine the stability behavior of the equilibrium solutions $P_1$, $P_2$, $P_3$, and $P_4$, and hence understand the phase portraits associated to the system (11).

To demonstrate the case $\beta\gamma > 1$, consider the following system in $x$-$y$ coordinates that we have shifted into $u$-$v$ coordinates for simplicity:

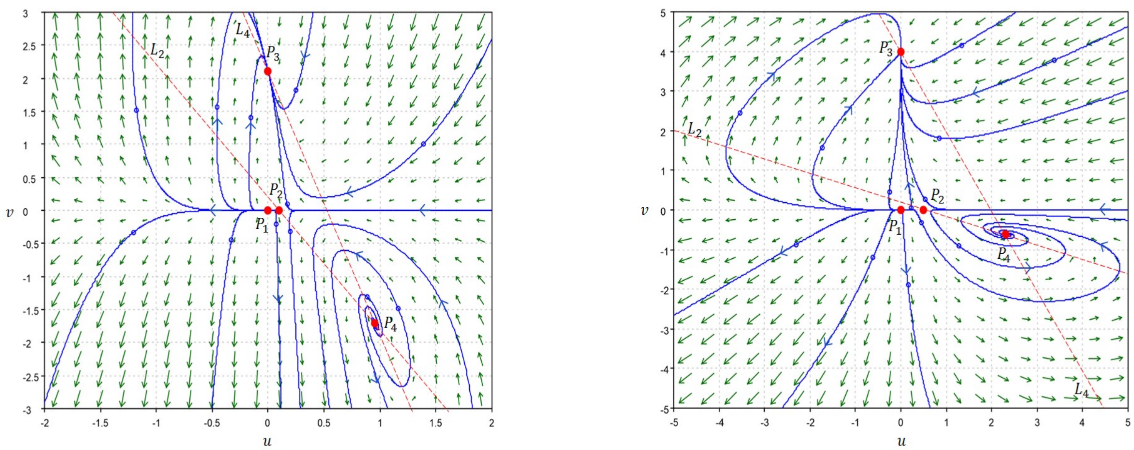
$$\begin{aligned} x' &= x(0.1 - 2x - y), \\ y' &= y(2.1 - 8x - 2y), \end{aligned} \quad \text{or rather,} \quad \begin{aligned} u' &= u(0.1 - u - 0.5v), \\ v' &= v(2.1 - 4u - v). \end{aligned}$$

Looking at the system in $u$-$v$ coordinates, we can see that $\alpha_1 = 0.1$, $\alpha_2 = 2.1$, $\beta = 4$, and $\gamma = 0.5$. Thus, $c_1 < 0 < c_2$. We can also compute $m$, $m^{-1}$, and $|c_2|/|c_1|$ to find that we have $1 < |c_2|/|c_1| = 34/19 < 5.83 \approx m$. Thus, from Table 3 we can see that $P_4$ is described by region $III$ of the $c_1$-$c_2$ plane, and from Table 4 we can see that $P_4$ will be an outward spiral. We can also use the relationship $c_1 < 0 < c_2$ and Table 1 to observe that $P_1$ will be a source, $P_2$ will be a saddle, and $P_3$ will be a sink. To verify our analysis, a phase portrait of the system in $u$-$v$ coordinates is shown in Figure 6a.

We give another example: consider a collection of parameters for equation (11) such that $\beta\gamma > 1$ and $(c_1, c_2)$ lies in region V of the $c_1$-$c_2$ plane, which is pictured in Figure 4. From Table 1, using that $c_1 < 0 < c_2$ when $(c_1, c_2)$ is in region V, we can conclude that $P_1$ is a source, $P_2$ is a saddle, and $P_3$ is a sink. Also, using Table 4, we can conclude that $P_4$ is asymptotically stable whose linearized solution is an inward spiral. This situation is pictured in Figure 6b for a particular choice of parameters such that $\beta\gamma > 1$ and $(c_1, c_2)$ lies in V.

Similar analysis can be done to construct the phase planes for solutions of equation (11) when $(c_1, c_2)$ lies in any of the viable regions of the $c_1$-$c_2$, that is whenever $(c_1, c_2)$ lies in one of the regions II through XI (not in the first quadrant, as that is not possible when $\beta\gamma > 1$). Thus, from our analysis, one can approach this model and, using only the parameters, fully understand the behaviors of solutions near equilibria in most scenarios (again, excluding centers).

Figure 6




(a) Numerical simulation of the $u$-$v$ plane when $(c_1, c_2)$ is in region III and $\beta\gamma > 1$.

(b) Numerical simulation of the $u$-$v$ plane when $(c_1, c_2)$ is in region V and $\beta\gamma > 1$.

# 6 Conclusion

This last section on phase-portrait characterizations demonstrates the power of linearization techniques. The systems of nonlinear ODEs discussed in the last section are difficult to solve, however, through the use of the techniques in this paper, we can simplify the problems and understand the general behaviors of the solutions. This process involves finding the equilibria of the nonlinear system, then linearizing the nonlinear system by taking the total derivative at the critical points. The next step was to use the linearized system to classify each equilibrium as stable or unstable based upon the eigenvalues of the coefficient matrix $A$. In the end, we arrived at a relatively simple characterization of the systems based on the values of their parameters.

Without a doubt, it would be preferable to find explicit solutions to nonlinear systems of ODEs, but finding such solutions can be very difficult (if not impossible) in general. Using linearization and stability analysis to construct qualitative solutions in phase planes is a viable alternative to searching for explicit solutions, as demonstrated here. Understanding the types of critical points that exist for a given system allowed us to draw phase portraits and visually express the behaviors of the solutions. We then applied these techniques to real-world models and characterized how the parameters of each system influenced the behaviors of the solutions. In this way, we developed a qualitative characterization of solutions to the equations we considered based only on the equation parameters and given initial conditions.

As a closing remark, this type of linear analysis of nonlinear problems is an accurate method of understanding the behaviors of solutions in the nonlinear system–with a few exceptions that were mentioned earlier. Methods of linearization become very useful when dealing with complicated systems that could be difficult or even impossible to solve, so it is beneficial to have a firm understanding of how to linearize and analyze real-world modeling problems as we did in this paper. Linearization is not appropriate in all situations, and it seems to run into trouble when the total derivative does not behave nicely, that is, if $DF(x)$ is not continuous or if it is not invertible at the critical point of interest. This situation manifested itself in our analysis when our equation had non-isolated equilibria.

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