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Matteo Copelli

Department of Mathematics and Statistics, University of Ottawa

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Department of Mathematics

Terre Haute, IN 47803

Email: mathjournal@rose-hulman.edu

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^aDepartment of Mathematics and Statistics, University of Ottawa

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CATEGORIFICATION OF THE NONNEGATIVE RATIONAL NUMBERS

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Abstract. In this document we describe a categorification of the semiring of natural numbers. We then use this result to construct a categorification of the semiring of nonnegative rational numbers.

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1 Introduction

Over the past few decades, the application of category theory to other fields of mathematics, such as representation theory, has experienced tremendous growth. Many algebraic concepts have been framed in category theoretic language and this new point of view has led to deeper understanding of representation theoretic objects. One prominent example is recent work on the categorification of quantum groups. The term “categorification” was introduced by Louis Crane and Igor Frenkel approximately 20 years ago. Since then, the categorification of algebraic and topological constructions has led to major advances in these fields. The purpose of the current paper is to give an example of categorification accessible to advanced undergraduate and beginning graduate students.

In this document, we categorify the semiring of nonnegative rational numbers. This semiring can be defined by taking the cartesian product of the natural numbers and the positive natural numbers, modulo a certain equivalence relation, and defining addition and multiplication operations appropriately. We mimic this construction on the level of categories. The first task we undertake is to categorify the natural numbers. Categorifying the natural numbers is one of the most commonly used examples of categorification. In Section 3, we describe a bijection from the set of isomorphism classes of the category of finitely generated vector spaces to the natural numbers. This bijection relies on the well-known result that two finitely generated vector spaces are isomorphic if and only if they have the same dimension.

Having completed this preliminary step, we then define another category. In Section 4, we create a categorification of $\mathbb{N} \times \mathbb{N}_+$. We call this category \mathcal{Q}' . We define a relation, that we call \sim , on the set of isomorphism classes of \mathcal{Q}' . We then construct an isomorphism from the set of equivalence classes of this relation to the set of nonnegative rational numbers and define operations on the set of equivalence classes. These operations correspond to addition and multiplication in the ring of the nonnegative rational numbers under our isomorphism. Finally we construct two formal morphisms that act as substitutes to re-enact the behaviour of \sim , but on the level of morphisms. We add these formal morphisms to the morphism class of \mathcal{Q}' to construct a bona fide categorification \mathcal{Q} of the nonnegative rational numbers.

Prerequisites. This document was written as an Undergraduate Honours Project at the University of Ottawa. It should therefore be accessible to most advanced undergraduate and beginning graduate mathematics students. Little to no knowledge of category theory or categorification is required, but knowledge of group and ring theory and advanced linear algebra is recommended.

2 Category Theory Review

In this section we review the concepts in category theory necessary to achieve the results we seek to prove. We also present formal definitions of some basic concepts related to the categorification of sets and semirings.

Definition 2.1 (Category). A *category* \mathcal{C} consists of a class of *objects* and a class of *morphisms*, denoted by $\text{Ob } \mathcal{C}$ and $\text{Mor } \mathcal{C}$ respectively. With these two classes we also include four assignments:

- Domain and codomain: For each morphism f there are given objects $\text{dom}(f)$ and $\text{codom}(f)$ called the *domain* and the *codomain* of f . We write $f: A \rightarrow B$ to denote that $A = \text{dom}(f)$ and that $B = \text{codom}(f)$.
- Identity: For each object A there is a given morphism $1_A: A \rightarrow A$ called the *identity morphism*.
- Composition: For morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ there is a given morphism $g \circ f: A \rightarrow C$ that is called the *composite of f and g* .

For these items to form a category, they are required to satisfy the following properties:

- Composition is associative.
- The identity morphism acts as a unit with regards to composition. I.e.:

$$f \circ 1_A = f = 1_B \circ f,$$

for all morphisms $f: A \rightarrow B$.

Definition 2.2 (Small category). A category \mathcal{C} is called a *small category* if $\text{Ob } \mathcal{C}$ and $\text{Mor } \mathcal{C}$ are both sets.

Definition 2.3 (Morphism class). Let \mathcal{C} be a category and let $A, B \in \text{Ob } \mathcal{C}$. We define the *morphism class* from A to B to be

$$\text{Mor}_{\mathcal{C}}(A, B) := \{f \in \text{Mor } \mathcal{C} \mid \text{dom}(f) = A \text{ and } \text{codom}(f) = B\}.$$

When the category in question is clear from the context, we will write $\text{Mor}(A, B)$ instead of $\text{Mor}_{\mathcal{C}}(A, B)$.

Definition 2.4 (Locally small category). Let \mathcal{C} be a category. If $\text{Mor}(A, B)$ is a set for all $A, B \in \text{Ob } \mathcal{C}$ then we say that \mathcal{C} is a *locally small category*.

Remark 2.5. In this article we will only be dealing with small (or locally small) categories. Thus, for the remainder of this document we will omit the terms “small” and “locally small” and simply refer to these as categories.

Example 2.6. Let \mathbb{K} be a field. Finitely generated vector spaces over \mathbb{K} form a category. The set $\text{Ob } \mathcal{C}$ is the set of finitely generated vector spaces and, for two finitely generated vector spaces V and W , $\text{Mor}(V, W)$ is the set of linear maps from V to W . This category is denoted by **FinVect** $_{\mathbb{K}}$.

Definition 2.7 (Subcategory, full subcategory). Let \mathcal{C} be a category. We call \mathcal{D} a *subcategory* of \mathcal{C} if $\text{Ob } \mathcal{D}$ is a subset of $\text{Ob } \mathcal{C}$ and $\text{Mor } \mathcal{D}$ is a subset of $\text{Mor } \mathcal{C}$. Additionally, for every $A \in \text{Ob } \mathcal{D}$ we have that $1_A \in \text{Mor } \mathcal{D}$ and for every $f \in \text{Mor } \mathcal{D}$ we have that $\text{dom}(f)$ and $\text{codom}(f) \in \text{Ob } \mathcal{D}$. Lastly, if $f, g \in \text{Mor } \mathcal{D}$ and $f \circ g$ is defined, then $f \circ g \in \text{Mor } \mathcal{D}$ (where \circ denotes composition in \mathcal{C}).

We call \mathcal{D} a *full subcategory* of \mathcal{C} if it is a subcategory of \mathcal{C} and, for every $A, B \in \text{Ob } \mathcal{D}$,

$$\text{Mor}_{\mathcal{D}}(A, B) = \text{Mor}_{\mathcal{C}}(A, B).$$

Definition 2.8 (Product category). The product of two categories \mathcal{C} and \mathcal{D} , denoted by $\mathcal{C} \times \mathcal{D}$, is a category. We call it the *product category* of \mathcal{C} and \mathcal{D} . The objects of this category are pairs of objects (A, B) where $A \in \text{Ob } \mathcal{C}$ and $B \in \text{Ob } \mathcal{D}$. If $(A_1, B_1), (A_2, B_2) \in \mathcal{C} \times \mathcal{D}$, the morphisms from (A_1, B_1) to (A_2, B_2) are pairs of morphisms (f, g) , where $f \in \text{Mor}_{\mathcal{C}}(A_1, A_2)$ and $g \in \text{Mor}_{\mathcal{D}}(B_1, B_2)$. Composition is component-wise. The identity morphism of the object (A, B) is $(1_A, 1_B)$.

Definition 2.9 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* F from \mathcal{C} to \mathcal{D} is a mapping such that:

- each $A \in \text{Ob } \mathcal{C}$ gets associated to an object $F(A) \in \mathcal{D}$,
- each $f \in \text{Mor}_{\mathcal{C}}(A, B)$ gets associated to a morphism $F(f) \in \text{Mor}_{\mathcal{D}}(F(A), F(B))$ such that the following two conditions hold:
 - For every $A \in \text{Ob } \mathcal{C}$ we have that $F(1_A) = 1_{F(A)}$.
 - Let $A, B, C \in \mathcal{C}$ and let $f \in \text{Mor}(A, B), g \in \text{Mor}(B, C)$. We have that $F(g \circ f) = F(g) \circ F(f)$.

Definition 2.10 (Bifunctor). Let \mathcal{C}, \mathcal{D} and \mathcal{E} be categories, a functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ is called a *bifunctor* (short for binary functor) from $\mathcal{C} \times \mathcal{D}$ to \mathcal{E} .

Definition 2.11 (Monoidal category). A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, e, \alpha, \lambda, \varrho)$ consisting of a category \mathcal{C} , a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an element e , and three natural isomorphisms α, λ , and ϱ that satisfy the following properties:

- For $A, B, C \in \text{Ob } \mathcal{C}$, we have that $\alpha_{A,B,C}: A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$.
- For $A \in \text{Ob } \mathcal{C}$, we have that $\lambda_A: e \otimes A \xrightarrow{\cong} A$ and that $\varrho_A: A \otimes e \xrightarrow{\cong} A$.

For this sextuple to be considered a monoidal category, these natural isomorphisms must satisfy the following two axioms:

- Pentagon axiom: For $A, B, C, D \in \text{Ob } \mathcal{C}$ the following pentagon commutes:

$$\begin{array}{ccc}
 & (A \otimes B) \otimes (C \otimes D) & \\
 \alpha_{A \otimes B, C, D} \nearrow & & \searrow \alpha_{A, B, C \otimes D} \\
 ((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\
 \alpha_{A, B, C} \otimes 1_D \searrow & & \nearrow 1_A \otimes \alpha_{B, C, D} \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D)
 \end{array}$$

- Triangle axiom: For $A, B \in \text{Ob } \mathcal{C}$ the following triangle commutes:

$$\begin{array}{ccc}
 (A \otimes e) \otimes B & \xrightarrow{\alpha_{A, e, B}} & A \otimes (e \otimes B) \\
 \varrho_A \otimes 1_B \searrow & & \nearrow 1_A \otimes \lambda_B \\
 & A \otimes B &
 \end{array}$$

Put simply, a monoidal category allows us to write expressions of the form $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ without any concern for placement of parentheses. All possible arrangements of parentheses are equivalent up to natural isomorphisms.

Example 2.12. An example that is pertinent to this article is the monoidal category $(\mathbf{FinVect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K}, \alpha, \lambda, \varrho)$, where $\otimes_{\mathbb{K}}$ denotes the usual tensor product of two vector spaces and the field \mathbb{K} is the identity element e . For any $U, V, W \in \mathbf{FinVect}_{\mathbb{K}}$, we define α to be the isomorphism

$$\begin{aligned}
 \alpha: U \otimes_{\mathbb{K}} (V \otimes_{\mathbb{K}} W) &\xrightarrow{\cong} (U \otimes_{\mathbb{K}} V) \otimes_{\mathbb{K}} W, \\
 u \otimes_{\mathbb{K}} (v \otimes_{\mathbb{K}} w) &\mapsto (u \otimes_{\mathbb{K}} v) \otimes_{\mathbb{K}} w, \text{ for all } u \in U, v \in V, w \in W.
 \end{aligned}$$

Let $k \in \mathbb{K}$. We define λ and ϱ to be the obvious isomorphisms:

$$\begin{aligned}
 \lambda: \mathbb{K} \otimes_{\mathbb{K}} V &\xrightarrow{\cong} V, & \varrho: V \otimes_{\mathbb{K}} \mathbb{K} &\xrightarrow{\cong} V, \\
 k \otimes_{\mathbb{K}} v &\mapsto kv, & v \otimes_{\mathbb{K}} k &\mapsto kv.
 \end{aligned}$$

Remark 2.13. In Example 2.12 we can see that α , λ and ϱ are pretty clear from the context. Henceforth, if these natural isomorphism are clear from the context of the monoidal category in discussion, we will omit them from our notation and simply refer to $(\mathcal{C}, \otimes, e)$ as the monoidal category.

Definition 2.14 (Zero object). Let \mathcal{C} be a category and let $X \in \text{Ob } \mathcal{C}$. We call X a *zero object* if, for every object A in the category, there exists exactly one morphism $X \rightarrow A$ and one morphism $A \rightarrow X$.

Example 2.15. In the category $\mathbf{FinVect}_{\mathbb{K}}$, the vector space $\{0\}$ is a zero object. This is because, for all $V \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}$, the only linear map from $\{0\} \rightarrow V$ is the map that sends 0 to 0_V and the only linear map from $V \rightarrow \{0\}$ is the map that sends all the elements of V to 0.

Definition 2.16 (Binary product). Let \mathcal{C} be a category and let $A, B \in \text{Ob } \mathcal{C}$. We call an object the *binary product* (this can be shortened to biproduct) of A and B , which we will denote as $A \amalg B$, if there exist morphisms $p_A: A \amalg B \rightarrow A$ and $p_B: A \amalg B \rightarrow B$ such that, for every $C \in \text{Ob } \mathcal{C}$ and every pair of morphisms f_A and f_B , the following diagram commutes for a unique morphism f :

$$\begin{array}{ccccc}
 & & C & & \\
 & f_A \swarrow & \vdots & \searrow f_B & \\
 & & f & & \\
 & \swarrow & \vdots & \searrow & \\
 A & \xleftarrow{p_A} & A \amalg B & \xrightarrow{p_B} & B
 \end{array}$$

Definition 2.17 (Additive category). We say that a category \mathcal{C} is an *additive category* if it satisfies the three following conditions:

- It has a zero object.
- Every morphism class is equipped with an addition (which we will denote as $+$ unless stated otherwise) that gives the morphism class the structure of an abelian group. Furthermore, composition of morphisms is distributive over $+$. In other words, for $f, f' \in \text{Mor}(A, B)$ and $g, g' \in \text{Mor}(B, C)$, we have:
 - a. $(g + g') \circ f = g \circ f + g' \circ f$,
 - b. $g \circ (f + f') = g \circ f + g \circ f'$.
- All finite biproducts of elements of the category exist.

Example 2.18. Let \mathbb{K} be a field. The category $\mathbf{FinVect}_{\mathbb{K}}$ is an additive category. The vector space $\{0\}$ is the zero object of $\mathbf{FinVect}_{\mathbb{K}}$, as we saw in Example 2.15. The direct sum of vector spaces (which will be denoted by \oplus) is the biproduct.

Remark 2.19. For further details concerning categories and a larger range of examples, we refer the reader to the texts by MacLane [ML98] or Awodey [Awo06].

Definition 2.20 (Semiring). A *semiring* is a quintuple $(X, \cdot, +, 1, 0)$, where X is a set equipped with two binary operations “ $+$ ” and “ \cdot ”, called addition and multiplication. It satisfies the following properties:

- (X, \cdot) is a monoid with identity element 1.
- $(X, +)$ is a commutative monoid with identity element 0.

- Multiplication is distributive over addition.
- Multiplication by 0 annihilates X . In other words, $0 \cdot x = x \cdot 0 = 0$ for all $x \in X$.

Definition 2.21 (Set of isomorphism classes). Let \mathcal{C} be a category and let \cong denote the relation of isomorphism in \mathcal{C} . We define $S^{\text{iso}}(\mathcal{C})$ to be *the set of isomorphism classes of the objects of \mathcal{C}* , i.e.:

$$S^{\text{iso}}(\mathcal{C}) := \text{Ob } \mathcal{C} / \cong.$$

We will use $[a]$ to denote the isomorphism class of $a \in \text{Ob } \mathcal{C}$ unless noted otherwise.

Definition 2.22 (Categorification of a set). To *categorify* a set A is to find a category \mathcal{C} and a bijection

$$\alpha: S^{\text{iso}}(\mathcal{C}) \rightarrow A.$$

We say that \mathcal{C} is a *categorification* of the set A .

Remark 2.23. Certain definitions of categorification do not require α to be a bijection. Some definitions have no restrictions on α . In the current paper, we are only interested in the case where α is indeed a bijection. Therefore, we will adopt the more rigorous definition stated above in Definition 2.22.

Definition 2.24 (Categorification of a semiring). Fix a semiring $(A, \cdot, +, 1, 0)$. Let $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $\oplus: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be bifunctors and let $I, O \in \text{Ob } \mathcal{C}$. We call $(\mathcal{C}, \otimes, \oplus, I, O)$ a *categorification of the semiring $(A, \cdot, +, 1, 0)$* if there exists a bijection

$$\alpha: S^{\text{iso}}(\mathcal{C}) \rightarrow A$$

that satisfies the following properties:

- For all $X, Y \in \text{Ob } \mathcal{C}$, we have $\alpha([X] \otimes [Y]) = \alpha([X]) \cdot \alpha([Y])$.
- For all $X, Y \in \text{Ob } \mathcal{C}$, we have $\alpha([X] \oplus [Y]) = \alpha([X]) + \alpha([Y])$.
- $\alpha(I) = 1$.
- $\alpha(O) = 0$.

For more details concerning categorification and its applications, we refer the reader to the texts by Savage [Sav] and Mazorchuk [Maz12] and the references therein.

3 Categorification of the Natural Numbers

In this section we categorify the semiring of natural numbers. We use the key fact that two finitely generated vector spaces are isomorphic if, and only if, they have the same dimension. This motivates us to define the function α of Definition 2.22 to be the dimension function (denoted as “dim”). We prove that dim induces a well-defined function on the set of

isomorphism classes of finitely generated vector spaces. This results in the categorification of \mathbb{N} as a set. We then show that the tensor product and direct sum of finitely generated vector spaces induce well-defined operations on $S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}})$ and that these operations correspond to addition and multiplication of natural numbers. Throughout this section, \mathbb{K} will denote a field.

Lemma 3.1. *The function $\dim: \text{Ob } \mathbf{FinVect}_{\mathbb{K}} \rightarrow \mathbb{N}$ is surjective.*

Proof. For all $n \in \mathbb{N}$, we have $\dim(\mathbb{K}^n) = n$. Thus \dim is surjective. \square

Lemma 3.2. *Let $f: A \rightarrow X$ be a surjective function and let \sim be the equivalence relation on A where we define $a \sim b \iff f(a) = f(b)$, for a and b elements of A . Then, f induces a new function, defined as follows:*

$$\begin{aligned} \bar{f}: A/\sim &\longrightarrow X, \\ [a] &\longmapsto f(a). \end{aligned}$$

The map \bar{f} is a bijection.

Proof. Let $f: A \rightarrow X$ be a surjective function and let \sim be the equivalence relation defined in the statement of the lemma.

- We will begin by demonstrating that \bar{f} is well defined. Let $a, b \in A$. Suppose that $[a] = [b]$, it follows that:

$$a \sim b \implies f(a) = f(b) \implies \bar{f}([a]) = \bar{f}([b]).$$

Therefore, \bar{f} is indeed well defined.

- Let us verify surjectivity. For all $x \in X$, there exists an $a \in A$ such that $f(a) = x$ by surjectivity of f . This implies that $\bar{f}([a]) = x$, thus, \bar{f} is surjective.
- We will now address injectivity. We have:

$$\bar{f}([a]) = \bar{f}([b]) \implies f(a) = f(b) \implies a \sim b \implies [a] = [b].$$

Thus, \bar{f} is an injective function.

Hence, \bar{f} is bijective. \square

Proposition 3.3. *The set of isomorphism classes of finitely generated vector spaces is isomorphic to the set of natural numbers, i.e. we have an isomorphism of sets:*

$$S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}}) \cong \mathbb{N}.$$

In other words, $\mathbf{FinVect}_{\mathbb{K}}$ is a categorification of the set \mathbb{N} .

Proof. Consider $\dim: \text{Ob } \mathbf{FinVect}_{\mathbb{K}} \rightarrow \mathbb{N}$. We have that \dim is surjective by Lemma 3.1. We also have that $\overline{\dim}: \mathbf{FinVect}_{\mathbb{K}}/\sim \rightarrow \mathbb{N}$ is a bijection by Lemma 3.2. For $M, N \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}$, we have:

$$M \sim N \iff \overline{\dim}([M]) = \overline{\dim}([N]) \iff \dim(M) = \dim(N) \iff M \cong N.$$

This is because two finite-dimensional vector spaces are isomorphic if, and only if, they have the same dimension. Therefore,

$$\begin{aligned} \overline{\dim}: \mathbf{FinVect}_{\mathbb{K}}/\cong &\longrightarrow \mathbb{N}, \\ [V] &\longmapsto \dim(V), \end{aligned}$$

is a well-defined bijection. □

Lemma 3.4. *Let $(\mathcal{C}, \otimes, e)$ be a monoidal category and let A and B be elements of $\text{Ob}(\mathcal{C})$. Then,*

$$[A] \otimes [B] := [A \otimes B]$$

is a well-defined operation on $S^{\text{iso}}(\mathcal{C})$.

Proof. Let $(\mathcal{C}, \otimes, e)$ be a monoidal category and let $A, B, A', B' \in \text{Ob } \mathcal{C}$. Suppose that $[A] = [A']$ and $[B] = [B']$. Then we have the following:

$$[A] \otimes [B] = [A \otimes B] = [A' \otimes B'] = [A'] \otimes [B'],$$

where, in the second equality, we use the fact that there exist isomorphisms $\phi: A \rightarrow A'$ and $\psi: B \rightarrow B'$. Thus, there is an isomorphism $\phi \otimes \psi: A \otimes B \rightarrow A' \otimes B'$. □

Lemma 3.5. *Let $(\mathcal{C}, \otimes, e)$ be a monoidal category. Then $(S^{\text{iso}}(\mathcal{C}), \otimes, [e])$ is a monoid.*

Proof. Let $(\mathcal{C}, \otimes, e, \alpha, \lambda, \varrho)$ be a monoidal category. Fix $A, B, C \in \text{Ob } \mathcal{C}$, we have:

$$\begin{aligned} [A] \otimes ([B] \otimes [C]) &= [A] \otimes ([B \otimes C]) \\ &= [A \otimes (B \otimes C)] \\ &= [\alpha(A \otimes (B \otimes C))] \\ &= [(A \otimes B) \otimes C] \\ &= ([A \otimes B]) \otimes [C] \\ &= ([A] \otimes [B]) \otimes [C], \end{aligned}$$

where, in the third equality above, we have used that α is an isomorphism. We also have:

$$[e] \otimes [A] = [e \otimes A] = [\lambda(e \otimes A)] = [A] = [\varrho(e \otimes A)] = [A \otimes e] = [A] \otimes [e],$$

where, in the second and fourth equality above, we used that λ and ϱ respectively are isomorphisms. Therefore, $(S^{\text{iso}}(\mathcal{C}), \otimes, [e])$ is a monoid. □

Lemma 3.6. *The map*

$$[V] \oplus [W] := [V \oplus W], \quad V, W \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}},$$

is a well-defined operation on $S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}})$, and $(S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}}), \oplus, \{0\})$ is a commutative monoid.

Proof. Let $U, V, W, U', V' \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}$. Suppose that $[U] = [U']$ and $[V] = [V']$. Then we have:

$$[U] \oplus [V] = [U \oplus V] = [U' \oplus V'] = [U'] \oplus [V'],$$

where, in the second equality, we use the fact that there exist isomorphisms $\phi: U \rightarrow U'$ and $\psi: V \rightarrow V'$. Thus, there is an isomorphism $\phi \oplus \psi: U \oplus V \rightarrow U' \oplus V'$. Therefore, \oplus is well defined on $S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}})$. It remains to show that we obtain the structure of a commutative monoid. To show associativity consider:

$$\begin{aligned} [U] \oplus ([V] \oplus [W]) &= [U] \oplus ([V \oplus W]) \\ &= [U \oplus (V \oplus W)] \\ &= [(U \oplus V) \oplus W] \\ &= ([U \oplus V]) \oplus [W] \\ &= ([U] \oplus [V]) \oplus [W], \end{aligned}$$

where, in the third equality, we use that direct sums of vector spaces are associative up to isomorphism. We also have that:

$$[\{0\}] \oplus [V] = [\{0\} \oplus V] = [V] = [V \oplus \{0\}] = [V] \oplus [\{0\}].$$

Lastly we need to show that \oplus is commutative. By definition we have that $[V] \oplus [W] = [V \oplus W] = [W \oplus V] = [W] \oplus [V]$, where we obtain the second equality because there exists an isomorphism $\gamma: V \oplus W \rightarrow W \oplus V$ defined by $\gamma(v \oplus w) = (w \oplus v)$. Thus, $(S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}}), \oplus, \{0\})$ is a commutative monoid. \square

Theorem 3.7. *We have that $(\mathbf{FinVect}_{\mathbb{K}}, \otimes, \oplus, \mathbb{K}, \{0\})$ is a categorification of the semiring $(\mathbb{N}, \cdot, +, 1, 0)$.*

Proof. Let $U, V \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}$. We have, as an isomorphism of sets, $S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}}) \cong \mathbb{N}$ by Proposition 3.3. We also saw in Example 2.12 that $(\mathbf{FinVect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ is a monoidal category. Thus, the operation $[U] \otimes [V] = [U \otimes V]$ is well defined on $S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}})$ by Lemma 3.4. Recall the definition of the induced function $\overline{\dim}$ used in Proposition 3.3. Then consider the following:

$$\overline{\dim}([U] \otimes [V]) = \overline{\dim}([U \otimes V]) = \dim(U \otimes V) = \dim(U) \cdot \dim(V) = \overline{\dim}([U]) \cdot \overline{\dim}([V]).$$

Similarly we have that $[U] \oplus [V] = [U \oplus V]$ is well defined on $S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}})$ by Lemma 3.6. Hence, we may consider:

$$\overline{\dim}([U] \oplus [V]) = \overline{\dim}([U \oplus V]) = \dim(U \oplus V) = \dim(U) + \dim(V) = \overline{\dim}([U]) + \overline{\dim}([V]).$$

Finally,

$$\begin{aligned}\overline{\dim}([\mathbb{K}]) &= \dim(\mathbb{K}) = 1, \\ \overline{\dim}([\{0\}]) &= \dim(\{0\}) = 0.\end{aligned}\quad \square$$

It should be noted that a categorification of the natural numbers can also be constructed by considering the category of finite sets. We refer the reader to a text by Baez and Dolan [BD98] for details.

4 Categorification of the Nonnegative Rational Numbers

Since the nonnegative natural numbers are constructed algebraically from the rational numbers, it is logical to ask if our categorification of the natural numbers in the previous section can be used to obtain a categorification of the semiring of nonnegative rational numbers. We begin by mimicking the construction of the nonnegative rational numbers as equivalence classes of pairs of natural numbers. That is, we define an equivalence relation on the set of isomorphism classes of an appropriate product category and then, inspired by the addition and multiplication of fractions, we define operations on the set of equivalence classes of this relation. Finally, we introduce formal morphisms that replicate the properties of the relation and add them to the class of morphisms of our product category. We then show that this new category yields a categorification of the semiring of nonnegative rational numbers. Throughout this section, \mathbb{K} is a field.

Definition 4.1 (The Category \mathcal{Q}'). Let \mathcal{Q}' denote the product category of $\mathbf{FinVect}_{\mathbb{K}}$ and $\mathbf{FinVect}_{\mathbb{K}^*}$, where we write $\mathbf{FinVect}_{\mathbb{K}^*}$ to denote the full subcategory of $\mathbf{FinVect}_{\mathbb{K}}$ whose objects are nonzero vector spaces. In other words, we have that \mathcal{Q}' has the following classes of objects and morphisms:

- $\text{Ob } \mathcal{Q}' = \text{Ob } \mathbf{FinVect}_{\mathbb{K}} \times \text{Ob } \mathbf{FinVect}_{\mathbb{K}^*}$.
- For $(V_1, W_1), (V_2, W_2) \in \text{Ob } \mathcal{Q}'$, we have that

$$\text{Mor}_{\mathcal{Q}'}((V_1, W_1), (V_2, W_2)) = \text{Mor}_{\mathbf{FinVect}_{\mathbb{K}}}(V_1, V_2) \times \text{Mor}_{\mathbf{FinVect}_{\mathbb{K}^*}}(W_1, W_2).$$

We also define composition of morphisms and the identity morphism as we would for a product category (as seen in Definition 2.8).

Lemma 4.2. *We have that $S^{\text{iso}}(\mathcal{Q}') \cong \mathbb{N} \times \mathbb{N}_+$ as sets.*

Proof. We have that:

$$S^{\text{iso}}(\mathcal{Q}') = S^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}} \times \mathbf{FinVect}_{\mathbb{K}^*}) = (\mathbf{FinVect}_{\mathbb{K}} \times \mathbf{FinVect}_{\mathbb{K}^*}) / \cong$$

$$\begin{aligned} &\cong (\mathbf{FinVect}_{\mathbb{K}}/\cong) \times (\mathbf{FinVect}_{\mathbb{K}^*}/\cong) \\ &\cong \mathbb{N} \times \mathbb{N}_+, \end{aligned}$$

where we get the last isomorphism by Proposition 3.3. \square

Lemma 4.3. *Let $(V_1, W_1), (V_2, W_2) \in \text{Ob } \mathcal{Q}'$. Then the relation:*

$$\begin{aligned} &(V_1, W_1) \sim (V_2, W_2) \\ \iff &(V_1 \otimes U_1, W_1 \otimes U_1) \cong (V_2 \otimes U_2, W_2 \otimes U_2), \text{ for some } U_1, U_2 \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}^*, \end{aligned}$$

is an equivalence relation.

Proof. Fix $(V_1, W_1), (V_2, W_2), (V_3, W_3) \in \text{Ob } \mathcal{Q}'$. Let \sim be the relation defined in the statement of the lemma. Then, for $U_1 = U_2 = \mathbb{K}$, we have:

$$(V_1 \otimes U_1, W_1 \otimes U_1) \cong (V_1 \otimes \mathbb{K}, W_1 \otimes \mathbb{K}) \cong (V_1 \otimes U_2, W_1 \otimes U_2) \implies (V_1, W_1) \sim (V_1, W_1).$$

Thus, \sim is reflexive.

Let $(V_1, W_1) \sim (V_2, W_2)$. Then there exists $U_1, U_2 \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}^*$ such that:

$$\begin{aligned} &(V_1 \otimes U_1, W_1 \otimes U_1) \cong (V_2 \otimes U_2, W_2 \otimes U_2) \\ \implies &(V_2 \otimes U_2, W_2 \otimes U_2) \cong (V_1 \otimes U_1, W_1 \otimes U_1) \\ \implies &(V_2, W_2) \sim (V_1, W_1). \end{aligned}$$

Thus, \sim is symmetric.

Let $(V_1, W_1) \sim (V_2, W_2)$ and $(V_2, W_2) \sim (V_3, W_3)$. Then there exists $U_1, U_2, U'_2, U_3 \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}^*$ such that:

$$(V_1 \otimes U_1, W_1 \otimes U_1) \cong (V_2 \otimes U_2, W_2 \otimes U_2) \text{ and } (V_2 \otimes U'_2, W_2 \otimes U'_2) \cong (V_3 \otimes U_3, W_3 \otimes U_3).$$

Fix $X_1 = U_1 \otimes U'_2$ and $X_2 = U_3 \otimes U_2$. Then:

$$\begin{aligned} (V_1 \otimes X_1, W_1 \otimes X_1) &= (V_1 \otimes U_1 \otimes U'_2, W_1 \otimes U_1 \otimes U'_2) \\ &\cong (V_2 \otimes U_2 \otimes U'_2, W_2 \otimes U_2 \otimes U'_2) \\ &\cong (V_2 \otimes U'_2 \otimes U_2, W_2 \otimes U'_2 \otimes U_2) \\ &\cong (V_3 \otimes U_3 \otimes U_2, W_3 \otimes U_3 \otimes U_2) \\ &\cong (V_3 \otimes X_2, W_3 \otimes X_2) \\ \implies &(V_1, W_1) \sim (V_3, W_3). \end{aligned}$$

Thus, \sim is transitive. Therefore, \sim is an equivalence relation. \square

Lemma 4.4. *Let \sim be the equivalence relation defined in the statement of Lemma 4.3. Let $(V_1, W_1), (V'_1, W'_1), (V_2, W_2), (V'_2, W'_2) \in \text{Ob } \mathcal{Q}'$, such that $(V_1, W_1) \cong (V'_1, W'_1)$ and $(V_2, W_2) \cong (V'_2, W'_2)$. Then*

$$(V_1, W_1) \sim (V_2, W_2) \implies (V'_1, W'_1) \sim (V'_2, W'_2).$$

Proof. We have that:

$$\begin{aligned} (V_1, W_1) \sim (V_2, W_2) &\implies (V_1 \otimes U_1, W_1 \otimes U_1) \cong (V_2 \otimes U_2, W_2 \otimes U_2) \\ &\implies (V'_1 \otimes U_1, W'_1 \otimes U_1) \cong (V'_2 \otimes U_2, W'_2 \otimes U_2) \\ &\implies (V'_1, W'_1) \sim (V'_2, W'_2), \end{aligned}$$

for some $U_1, U_2 \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}^*$. □

Definition 4.5. The equivalence relation \sim in Lemma 4.3 induces a relation \sim on $S^{\text{iso}}(\mathcal{Q}')$. More precisely:

$$[(V_1, W_1)] \sim [(V_2, W_2)] \iff (V_1, W_1) \sim (V_2, W_2), \quad \text{for all } (V_1, W_1), (V_2, W_2) \in \text{Ob } \mathcal{Q}'.$$

This is well defined by Lemma 4.4. We write $\overline{[V_1, W_1]}$ to denote the equivalence class of $[(V_1, W_1)]$ under the relation \sim .

Proposition 4.6. *The map*

$$\begin{aligned} \eta: S^{\text{iso}}(\mathcal{Q}')/\sim &\rightarrow \mathbb{Q}_{\geq} \\ \overline{[V, W]} &\mapsto \frac{\dim(V)}{\dim(W)}, \end{aligned}$$

is a bijection. In other words, we have that

$$S^{\text{iso}}(\mathcal{Q}')/\sim \cong \mathbb{Q}_{\geq},$$

as sets.

Proof. Let $(V_1, W_1), (V_2, W_2) \in \mathcal{Q}'$ and suppose that $\eta(\overline{[V_1, W_1]}) = \eta(\overline{[V_2, W_2]})$. We have that:

$$\begin{aligned} \eta(\overline{[V_1, W_1]}) &= \eta(\overline{[V_2, W_2]}) \\ \implies \frac{\dim(V_1)}{\dim(W_1)} &= \frac{\dim(V_2)}{\dim(W_2)}. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \frac{\dim(V_1) \dim(W_2)}{\dim(W_1) \dim(W_2)} &= \frac{\dim(V_2) \dim(W_1)}{\dim(W_2) \dim(W_1)} \\ \implies \frac{\dim(V_1 \otimes W_2)}{\dim(W_1 \otimes W_2)} &= \frac{\dim(V_2 \otimes W_1)}{\dim(W_2 \otimes W_1)}. \end{aligned}$$

This implies that $\dim(V_1 \otimes W_2) = \dim(V_2 \otimes W_1)$, since $\dim(W_1 \otimes W_2) = \dim(W_2 \otimes W_1)$. Thus,

$$(V_1 \otimes W_2) \cong (V_2 \otimes W_1) \text{ and obviously } (W_1 \otimes W_2) \cong (W_2 \otimes W_1)$$

$$\begin{aligned}
&\implies (V_1 \otimes W_2, W_1 \otimes W_2) \cong (V_2 \otimes W_1, W_2 \otimes W_1) \\
&\implies [(V_1, W_1)] \sim [(V_2, W_2)] \\
&\implies \overline{[V_1, W_1]} = \overline{[V_2, W_2]}.
\end{aligned}$$

Thus, η is injective.

For all $a \in \mathbb{N}$ and $b \in \mathbb{N}_+$, we have that $\eta\left(\overline{[\mathbb{K}^a, \mathbb{K}^b]}\right) = \frac{a}{b}$. Thus, η is surjective. Therefore, η is a bijection. \square

Lemma 4.7. *Let $(V_1, W_1), (V_2, W_2) \in \mathcal{Q}'$. The operation:*

$$\overline{[V_1, W_1]} \otimes \overline{[V_2, W_2]} := \overline{[V_1 \otimes V_2, W_1 \otimes W_2]},$$

is a well-defined operation on $S^{\text{iso}}(\mathcal{Q}')/\sim$.

Proof. Let $(V_1, W_1), (V_2, W_2), (V'_1, W'_1), (V'_2, W'_2) \in \mathcal{Q}'$ such that $\overline{[V_1, W_1]} = \overline{[V'_1, W'_1]}$ and $\overline{[V_2, W_2]} = \overline{[V'_2, W'_2]}$. We have:

$$\begin{aligned}
&\overline{[V_1, W_1]} = \overline{[V'_1, W'_1]} \\
&\implies [(V_1, W_1)] \sim [(V'_1, W'_1)] \\
&\implies (V_1, W_1) \sim (V'_1, W'_1) \\
&\implies (V_1 \otimes U_1, W_1 \otimes U_1) \cong (V'_1 \otimes U'_1, W'_1 \otimes U'_1) \text{ for some } U_1, U'_1 \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}^*.
\end{aligned}$$

Similarly, since $\overline{[V_2, W_2]} = \overline{[V'_2, W'_2]}$, we have that

$$(V_2 \otimes U_2, W_2 \otimes U_2) \cong (V'_2 \otimes U'_2, W'_2 \otimes U'_2) \text{ for some } U_2, U'_2 \in \text{Ob } \mathbf{FinVect}_{\mathbb{K}}^*.$$

Therefore,

$$(V_1 \otimes V_2 \otimes (U_1 \otimes U_2), W_1 \otimes W_2 \otimes (U_1 \otimes U_2)) \cong (V'_1 \otimes V'_2 \otimes (U'_1 \otimes U'_2), W'_1 \otimes W'_2 \otimes (U'_1 \otimes U'_2)).$$

Thus, we have that $[(V_1 \otimes V_2, W_1 \otimes W_2)] \sim [(V'_1 \otimes V'_2, W'_1 \otimes W'_2)]$. Using this, we have:

$$\begin{aligned}
\overline{[V_1, W_1]} \otimes \overline{[V_2, W_2]} &= \overline{[V_1 \otimes V_2, W_1 \otimes W_2]} \\
&= \overline{[V'_1 \otimes V'_2, W'_1 \otimes W'_2]} \\
&= \overline{[V'_1, W'_1]} \otimes \overline{[V'_2, W'_2]}.
\end{aligned}$$

Thus, \otimes is well defined on $S^{\text{iso}}(\mathcal{Q}')/\sim$. \square

Lemma 4.8. *Let $(V_1, W_1), (V_2, W_2) \in \mathcal{Q}'$. The operation:*

$$\overline{[V_1, W_1]} \oplus \overline{[V_2, W_2]} := \overline{[(V_1 \otimes W_2) \oplus (V_2 \otimes W_1), W_1 \otimes W_2]},$$

is a well-defined operation on $S^{\text{iso}}(\mathcal{Q}')/\sim$.

Proof. Let $(V_1, W_1), (V_2, W_2), (V'_1, W'_1), (V'_2, W'_2) \in \mathcal{Q}'$ such that $\overline{[V_1, W_1]} = \overline{[V'_1, W'_1]}$ and $\overline{[V_2, W_2]} = \overline{[V'_2, W'_2]}$. As in the proof for Lemma 4.7, we have:

$$\begin{aligned} (V_1 \otimes U_1, W_1 \otimes U_1) &\cong (V'_1 \otimes U'_1, W'_1 \otimes U'_1) \text{ for some } U_1, U_2 \in \mathbf{FinVect}_{\mathbb{K}}^*, \\ (V_2 \otimes U_2, W_2 \otimes U_2) &\cong (V'_2 \otimes U'_2, W'_2 \otimes U'_2) \text{ for some } U'_1, U'_2 \in \mathbf{FinVect}_{\mathbb{K}}^*. \end{aligned}$$

Therefore, we have that:

$$\begin{aligned} ((V_1 \otimes U_1 \otimes W_2 \otimes U_2) \oplus (V_2 \otimes U_2 \otimes W_1 \otimes U_1), W_1 \otimes U_1 \otimes W_2 \otimes U_2) \\ \cong ((V'_1 \otimes U'_1 \otimes W'_2 \otimes U'_2) \oplus (V'_2 \otimes U'_2 \otimes W'_1 \otimes U'_1), W'_1 \otimes U'_1 \otimes W'_2 \otimes U'_2). \end{aligned}$$

Thus,

$$\begin{aligned} (((V_1 \otimes W_2) \otimes (U_1 \otimes U_2)) \oplus ((V_2 \otimes W_1) \otimes (U_1 \otimes U_2)), (W_1 \otimes W_2) \otimes (U_1 \otimes U_2)) \\ \cong (((V'_1 \otimes W'_2) \otimes (U'_1 \otimes U'_2)) \oplus ((V'_2 \otimes W'_1) \otimes (U'_1 \otimes U'_2)), (W'_1 \otimes W'_2) \otimes (U'_1 \otimes U'_2)). \end{aligned}$$

This implies that,

$$\begin{aligned} (((V_1 \otimes W_2) \oplus (V_2 \otimes W_1)) \otimes (U_1 \otimes U_2), (W_1 \otimes W_2) \otimes (U_1 \otimes U_2)) \\ \cong (((V'_1 \otimes W'_2) \oplus (V'_2 \otimes W'_1)) \otimes (U'_1 \otimes U'_2), (W'_1 \otimes W'_2) \otimes (U'_1 \otimes U'_2)). \end{aligned}$$

Therefore,

$$[((V_1 \otimes W_2) \oplus (V_2 \otimes W_1), W_1 \otimes W_2)] \sim [((V'_1 \otimes W'_2) \oplus (V'_2 \otimes W'_1), W'_1 \otimes W'_2)].$$

Using this result we have that:

$$\begin{aligned} \overline{[V_1, W_1]} \oplus \overline{[V_2, W_2]} &= \overline{[(V_1 \otimes W_2) \oplus (V_2 \otimes W_1), W_1 \otimes W_2]} \\ &= \overline{[(V'_1 \otimes W'_2) \oplus (V'_2 \otimes W'_1), W'_1 \otimes W'_2]} \\ &= \overline{[V'_1, W'_1]} \oplus \overline{[V'_2, W'_2]}. \end{aligned} \quad \square$$

Theorem 4.9. *The isomorphism η from Proposition 4.6 has the following properties for all $(V_1, W_1), (V_2, W_2) \in \mathcal{Q}'$,:*

$$\begin{aligned} \eta(\overline{[V_1, W_1]} \otimes \overline{[V_2, W_2]}) &= \eta(\overline{[V_1, W_1]}) \cdot \eta(\overline{[V_2, W_2]}), \\ \eta(\overline{[V_1, W_1]} \oplus \overline{[V_2, W_2]}) &= \eta(\overline{[V_1, W_1]}) + \eta(\overline{[V_2, W_2]}), \\ \eta(\overline{[\mathbb{K}, \mathbb{K}]}) &= 1, \\ \eta(\overline{[\{0\}, \mathbb{K}]}) &= 0. \end{aligned}$$

Therefore η is an isomorphism of semirings between $(S^{\text{iso}}(\mathcal{Q}')/\sim, \otimes, \oplus, \overline{[\mathbb{K}, \mathbb{K}]}, \overline{[\{0\}, \mathbb{K}]})$ and $(\mathbb{Q}_{\geq}, \cdot, +, 1, 0)$.

Proof. We have that $S^{\text{iso}}(\mathcal{Q}')/\sim \cong \mathbb{Q}_{\geq}$ as sets by Proposition 4.6. We also saw in Lemma 4.7 that the operation $\overline{[V_1, W_1]} \otimes \overline{[V_2, W_2]} = \overline{[V_1 \otimes V_2, W_1 \otimes W_2]}$ is well defined on $S^{\text{iso}}(\mathcal{Q}')/\sim$. Recall the definition of the bijection η used in the proof of Proposition 4.6, then consider the following:

$$\begin{aligned} \eta\left(\overline{[V_1, W_1]} \otimes \overline{[V_2, W_2]}\right) &= \eta\left(\overline{[V_1 \otimes V_2, W_1 \otimes W_2]}\right) \\ &= \frac{\dim(V_1 \otimes V_2)}{\dim(W_1 \otimes W_2)} \\ &= \frac{\dim(V_1) \cdot \dim(V_2)}{\dim(W_1) \cdot \dim(W_2)} \\ &= \frac{\dim(V_1)}{\dim(W_1)} \cdot \frac{\dim(V_2)}{\dim(W_2)} \\ &= \eta\left(\overline{[V_1, W_1]}\right) \cdot \eta\left(\overline{[V_2, W_2]}\right). \end{aligned}$$

Similarly we have that $\overline{[V_1, W_1]} \oplus \overline{[V_2, W_2]} = \overline{[(V_1 \otimes W_2) \oplus (V_1 \otimes W_2), W_1 \otimes W_2]}$ is well defined on $S^{\text{iso}}(\mathcal{Q}')/\sim$ by Lemma 4.8. Therefore, we may consider,

$$\begin{aligned} \eta\left(\overline{[V_1, W_1]} \oplus \overline{[V_2, W_2]}\right) &= \eta\left(\overline{[(V_1 \otimes W_2) \oplus (V_1 \otimes W_2), W_1 \otimes W_2]}\right) \\ &= \frac{\dim((V_1 \otimes W_2) \oplus (V_1 \otimes W_2))}{\dim(W_1 \otimes W_2)} \\ &= \frac{\dim(V_1 \otimes W_2) + \dim(V_1 \otimes W_2)}{\dim(W_1 \otimes W_2)} \\ &= \frac{\dim(V_1) \cdot \dim(W_2) + \dim(V_1) \cdot \dim(W_2)}{\dim(W_1) \cdot \dim(W_2)} \\ &= \frac{\dim(V_1)}{\dim(W_1)} + \frac{\dim(V_2)}{\dim(W_2)} \\ &= \eta\left(\overline{[V_1, W_1]}\right) + \eta\left(\overline{[V_2, W_2]}\right). \end{aligned}$$

Finally, we have

$$\begin{aligned} \eta\left(\overline{[\mathbb{K}, \mathbb{K}]}\right) &= \frac{\dim(\mathbb{K})}{\dim(\mathbb{K})} = 1, \\ \eta\left(\overline{[\{0\}, \mathbb{K}]}\right) &= \frac{\dim(\{0\})}{\dim(\mathbb{K})} = 0. \end{aligned} \quad \square$$

Note that Theorem 4.9 does not, strictly speaking, give a categorification of the semiring $\mathbb{Q}_{\geq 0}$ since it is the set of equivalence classes of $S^{\text{iso}}(\mathcal{Q}')$, and not $S^{\text{iso}}(\mathcal{Q}')$ itself, that is isomorphic to $\mathbb{Q}_{\geq 0}$. In the remainder of the paper, we remedy this situation.

Definition 4.10 (Sew and tear morphisms). Let $(V, W) \in \mathcal{Q}'$ and let $U \in \mathbf{FinVect}_{\mathbb{K}}^*$. We define a *sew morphism of U and (V, W)* , denoted by $\varsigma_{V, W}^U$, to be a formal morphism with

domain (V, W) and codomain $(V \otimes U, W \otimes U)$. Alternatively, we define a *tear morphism of U and (V, W)* , denoted by $\tau_{V,W}^U$, to be a formal morphism with domain $(V \otimes U, W \otimes U)$ and codomain (V, W) . If the domain and codomain of the sew (or tear) morphism are clear from the context we will omit them from the notation.

Definition 4.11 (Alphabet set X). Let $U, W \in \mathbf{FinVect}_{\mathbb{K}}^*$ and let $V \in \mathbf{FinVect}_{\mathbb{K}}$. We define the *alphabet set*, denoted by X , to be:

$$X = \text{Mor}_{\mathcal{Q}} \cup \{\varsigma_{V,W}^U\}_{U,V,W} \cup \{\tau_{V,W}^U\}_{U,V,W}.$$

Definition 4.12 (Words in X). We define a *word* in X to be a sequence $a_1 a_2 \dots a_n$, where $a_1, a_2, \dots, a_n \in X$, such that $\text{dom}(a_i) = \text{codom}(a_{i-1})$ for $i = 2, 3, \dots, n$.

Definition 4.13 (Subwords). Let w be a word in X . Any subsequence of consecutive elements in the word w is called a *subword* of w . The reader should note that any subword is a word.

Definition 4.14 (Simplification relation, Simplification of a word). We say that w_2 is a *simplification of w_1* , and we write $w_1 \rightarrow w_2$, if one of the following holds:

- $w_1 = w'_1 f g w''_1$ and $w_2 = w'_1 (f \circ g) w''_1$, where f and g are linear transformations and w'_1 and w''_1 are subwords of w_1 .
- $w_1 = w'_1 \varsigma^U \varsigma^{U'} w''_1$ and $w_2 = w'_1 (\varsigma^{U \otimes U'}) w''_1$ for some $U, U' \in \mathbf{FinVect}_{\mathbb{K}}^*$.
- $w_1 = w'_1 \tau^U \tau^{U'} w''_1$ and $w_2 = w'_1 (\tau^{U \otimes U'}) w''_1$ for some $U, U' \in \mathbf{FinVect}_{\mathbb{K}}^*$.
- $w_1 = w'_1 \tau^U \varsigma^U w''_1$ and $w_2 = w'_1 1_{\text{dom}(\varsigma^U)} w''_1 = w'_1 w''_1$ for some $U \in \mathbf{FinVect}_{\mathbb{K}}^*$.
- $w_1 = w'_1 \varsigma^U \tau^U w''_1$ and $w_2 = w'_1 1_{\text{dom}(\tau^U)} w''_1 = w'_1 w''_1$ for some $U \in \mathbf{FinVect}_{\mathbb{K}}^*$.

We then consider the transitive closure of the relation \rightarrow , which we again denote by \rightarrow . That is, we write $w_1 \rightarrow w_2$ if w_2 can be obtained from w_1 via a sequence of the above simplifications. It is important to note that simplification results in a shorter word. Thus, if we have $w_1 \rightarrow w_2$, it implies the length of w_2 is less than the length of w_1 . Therefore, after a finite number of simplifications, we produce a word that can no longer be simplified. In other words, the process of simplification always terminates.

Definition 4.15 (Reduced words in X). We define a *reduced word* to be a word that cannot be simplified.

Definition 4.16 (The category \mathcal{Q}). Let \mathcal{Q} be the category with the following classes of objects and morphisms:

- $\text{Ob } \mathcal{Q} = \text{Ob } \mathcal{Q}'$.
- $\text{Mor } \mathcal{Q}$ is the set of all reduced words in X .

The identity morphism for (V, W) is the pair $(1_V, 1_W)$ (same as it was in \mathcal{Q}'). The identity can also be viewed as an *empty word*. The composition of two morphisms, w_1 and w_2 , is defined in the following way:

$$w_1 \circ w_2 = w_3,$$

where w_3 is the reduced word such that $w_1 w_2 \rightarrow w_3$. The verification that \mathcal{Q} is a category is straightforward and left as an exercise to the reader.

Lemma 4.17. *A word in $\text{Mor } \mathcal{Q}$ is an isomorphism if and only if it is a sequence of sew morphisms, tear morphisms and linear maps that are isomorphisms (in the usual sense).*

Proof. Fix $w \in \text{Mor } \mathcal{Q}$. Let $w = a_1 a_2 \dots a_n$ where $a_1, a_2, \dots, a_n \in X$. If a_1, a_2, \dots, a_n are sew morphisms, tear morphisms or invertible linear maps then the inverse of $a_1 a_2 \dots a_n$ is simply $a_n^{-1} a_{n-1}^{-1} \dots a_1^{-1}$. Hence, w is invertible.

It remains to prove the reverse implication. Suppose w is invertible. We will proceed by induction on the length of w . If w is of length zero, then it is empty and therefore all of its letters are invertible (as it has none). Now consider w to be of length n . In other words, let $w = a_1 a_2 \dots a_n$ be a reduced word. If a_n is a sew morphism, a tear morphism or an invertible linear map, then it has an inverse b . Then $w \circ b = a_1 a_2 \dots a_{n-1}$ is invertible (being the composition of invertible morphisms) and the result follows by the inductive hypothesis. Now suppose, towards a contradiction, that a_n is a linear map that is not invertible. Let $w^{-1} = b_1 b_2 \dots b_m$ be the inverse of w . Thus, we have

$$(a_1 a_2 \dots a_n)(b_1 b_2 \dots b_m) = 1_{\text{codom}(a_1)} \implies a_1 a_2 \dots (a_n b_1) b_2 \dots b_m = 1_{\text{codom}(a_1)}.$$

Since w and w^{-1} are elements of $\text{Mor } \mathcal{Q}$ they must be reduced words. Therefore, the only place possible for simplification (to arrive at the identity) is in the subword $a_n b_1$. Thus, b_1 must be a linear map and either

$$a_n b_1 = a_n \circ b_1 = 1_{\text{codom}(a_n)} \quad \text{or} \quad a_n b_1 = a_n \circ b_1 \neq 1_{\text{codom}(a_n)}.$$

In the first case, a_n is an invertible linear map, contradicting our assumption. In the second case, we have $a_n b_1 \neq 1_{\text{codom}(a_n)}$ and $a_1 \dots a_{n-1} (a_n \circ b_1) b_2 \dots b_m = 1_{\text{codom}(a_1)}$. This implies that a_{n-1} or b_2 must simplify with $a_n \circ b_1$. Thus, a_{n-1} or b_2 must be linear maps. This is a contradiction because w and w^{-1} are reduced words in X . This completes the proof. \square

Proposition 4.18. *Let $(V_1, W_1), (V_2, W_2) \in \text{Ob } \mathcal{Q}$. We have that:*

$$\langle (V_1, W_1) \rangle = \langle (V_2, W_2) \rangle \iff \overline{[V_1, W_1]} = \overline{[V_2, W_2]},$$

where $\langle V_1, W_1 \rangle$ is used to denote the isomorphism class of (V_1, W_1) in \mathcal{Q} .

Thus, we have a natural bijection of sets,

$$\begin{aligned} \varphi: S^{\text{iso}}(\mathcal{Q}) &\rightarrow S^{\text{iso}}(\mathcal{Q}')/\sim \\ \langle V_1, W_1 \rangle &\mapsto \overline{[V_1, W_1]}. \end{aligned}$$

Proof. Let $\overline{[V_1, W_1]} = \overline{[V_2, W_2]}$. Then:

$$[(V_1, W_1)] \sim [(V_2, W_2)] \implies (V_1, W_1) \sim (V_2, W_2).$$

Therefore, there exists $U, U' \in \mathbf{FinVect}^*$ such that:

$$(V_1 \otimes U, W_1 \otimes U) \cong_{\mathcal{Q}'} (V_2 \otimes U', W_2 \otimes U').$$

However, in \mathcal{Q} we have that,

$$(V_1, W_1) \cong_{\mathcal{Q}} (V_1 \otimes U, W_1 \otimes U) \quad \text{and} \quad (V_2, W_2) \cong_{\mathcal{Q}} (V_2 \otimes U', W_2 \otimes U').$$

Thus,

$$(V_1, W_1) \cong_{\mathcal{Q}} (V_2, W_2) \implies \langle V_1, W_1 \rangle = \langle V_2, W_2 \rangle.$$

This proves one direction of the dual implication.

To prove the reverse implication, suppose $\langle V_1, W_1 \rangle = \langle V_2, W_2 \rangle$. Then there exists a word w such that

$$w: (V_1, W_1) \xrightarrow{\cong} (V_2, W_2)$$

is an isomorphism in \mathcal{Q} . If the w is the identity morphism, the result holds trivially. Now suppose that w is of length one.

- If w is a sew morphism, then $w = \zeta^U$ for some $U \in \mathbf{FinVect}^*$. Thus,

$$(V_2, W_2) = (V_1 \otimes U, W_1 \otimes U),$$

which implies that $\overline{[V_1, W_1]} = \overline{[V_2, W_2]}$.

- If w is a tear morphism, then $w = \tau^U$ for some $U \in \mathbf{FinVect}^*$. Thus,

$$(V_1, W_1) = (V_2 \otimes U, W_2 \otimes U),$$

which implies that $\overline{[V_1, W_1]} = \overline{[V_2, W_2]}$.

- If w is an invertible linear map, then $V_1 \cong V_2$ and $W_1 \cong W_2$ as vector spaces. Thus, $\overline{[V_1, W_1]} = \overline{[V_2, W_2]}$.

Thus, the result is true for w of length one. Now suppose w is of length n . Let $w = a_n \dots a_2 a_1$. By Lemma 4.17, each a_i , $i = 1, \dots, n$, is either a sew morphism, a tear morphism, or an invertible linear map. Therefore, by the above, we have the chain of equalities:

$$\overline{[V_1, W_1]} = \overline{[\text{codom}(a_1)]} = \dots = \overline{[\text{codom}(a_{n-1})]} = \overline{[V_2, W_2]}. \quad \square$$

Definition 4.19 (The operations \otimes and \oplus on \mathcal{Q}). We define \otimes to be the following operation on $S^{\text{iso}}(\mathcal{Q})$:

$$\langle V_1, W_1 \rangle \otimes \langle V_2, W_2 \rangle = \langle V_1 \otimes V_2, W_1 \otimes W_2 \rangle, \quad \text{for all } (V_1, W_1), (V_2, W_2) \in \mathcal{Q}.$$

We also define \oplus to be:

$$\langle V_1, W_1 \rangle \oplus \langle V_2, W_2 \rangle = \langle (V_1 \otimes W_2) \oplus (V_2 \otimes W_1), W_1 \otimes W_2 \rangle.$$

These operations are well defined by Lemmas 4.7, 4.8 and Proposition 4.18.

Theorem 4.20. *We have that $(\mathcal{Q}, \otimes, \oplus, (\mathbb{K}, \mathbb{K}), (0, \mathbb{K}))$ is a categorification of the semiring $(\mathbb{Q}_{\geq}, \cdot, +, 1, 0)$.*

Proof. Let $(V_1, W_1), (V_2, W_2) \in \mathcal{Q}$. We have that $S^{\text{iso}}(\mathcal{Q}) \cong S^{\text{iso}}(\mathcal{Q}')/\sim$ as sets by the natural bijection φ from Proposition 4.18. We also have that:

$$\begin{aligned} \varphi(\langle V_1, W_1 \rangle \otimes \langle V_2, W_2 \rangle) &= \varphi(\langle V_1 \otimes V_2, W_1 \otimes W_2 \rangle) \\ &= \overline{[V_1 \otimes V_2, W_1 \otimes W_2]} \\ &= \overline{[V_1, W_1] \otimes [V_2, W_2]} \\ &= \varphi(\langle V_1, W_1 \rangle) \otimes \varphi(\langle V_2, W_2 \rangle). \end{aligned}$$

Similarly we have that:

$$\begin{aligned} \varphi(\langle V_1, W_1 \rangle \oplus \langle V_2, W_2 \rangle) &= \varphi(\langle (V_1 \otimes W_2) \oplus (V_2 \otimes W_1), W_1 \otimes W_2 \rangle) \\ &= \overline{[(V_1 \otimes W_2) \oplus (V_2 \otimes W_1), W_1 \otimes W_2]} \\ &= \overline{[V_1, W_1] \oplus [V_2, W_2]} \\ &= \varphi(\langle V_1, W_1 \rangle) \oplus \varphi(\langle V_2, W_2 \rangle). \end{aligned}$$

By definition, φ also maps

$$\langle 0, \mathbb{K} \rangle \mapsto \overline{[0, \mathbb{K}]} \quad \text{and} \quad \langle \mathbb{K}, \mathbb{K} \rangle \mapsto \overline{[\mathbb{K}, \mathbb{K}]}.$$

Recall the function η from Proposition 4.6. By the above and Theorem 4.9,

$$\eta \circ \varphi: S^{\text{iso}}(\mathcal{Q}) \rightarrow \mathbb{Q}_{\geq}$$

is an isomorphism of rings and therefore, $(\mathcal{Q}, \otimes, \oplus, (\mathbb{K}, \mathbb{K}), (0, \mathbb{K}))$ is a categorification of the semiring $(\mathbb{Q}_{\geq}, \cdot, \oplus, 1, 0)$. \square

References

- [Awo06] Steve Awodey. *Category theory*, volume 49 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 2006.
- [BD98] John C. Baez and James Dolan. Categorification. In *Higher category theory (Evanston, IL, 1997)*, volume 230 of *Contemp. Math.*, pages 1–36. Amer. Math. Soc., Providence, RI, 1998.
- [Maz12] Volodymyr Mazorchuk. *Lectures on algebraic categorification*. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012.
- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [Sav] Alistair Savage. Introduction to categorification. Preprint: [arXiv:1401.6037](https://arxiv.org/abs/1401.6037) [math.RT].