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Joshua Cape
Rhodes College

Willam Dearden
Lehigh University

William Gamber
Pomona College

Linh Nguyen
Lafayette College

Rose-
Hulman
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Joshua Cape[a] William Dearden[b]
William Gamber[c] Linh Nguyen[d]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Rhodes College
[b]Lehigh University
[c]Pomona College
[d]Lafayette College

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First Encounters with Option Pricing and Return Simulation

Joshua Cape William Dearden William Gamber Linh Nguyen

Abstract. We provide a tractable introduction to option pricing models and examine how the complex analysis concept of *branch-cutting* influences financial mathematics. The Black-Scholes model is introduced to motivate our discussion of the Heston stochastic volatility model, a model which dominates industry and option pricing literature in financial mathematics. We focus on developing mathematical intuition as a tool for stimulating further undergraduate interest and research in financial mathematics. We provide code in R and Mathematica for applications.

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1 Introduction

In 1993, Heston introduced a stochastic volatility model for pricing financial derivatives [11]. He offered two alternative approaches: a simulation method and a semi-analytic option pricing formula. A branch-cutting problem occurred within the original paper's semi-analytic option pricing formula, and this led to the mispricing of financial derivatives within the financial sector, derivatives which included call and put options. While a single such option can be explicitly priced using the simulation method, the semi-analytic formula tends to be computationally more efficient for modeling an investment strategy over a period of time–say investing monthly in options and rolling over the investment strategy for several years. Heston's original semi-analytic option pricing formula has since been re-examined and corrected [1].

Undergraduate textbooks in financial mathematics often ignore the Heston model because they wish to avoid measure theory ([3] [6] [9] [12] [14] [17]). Graduate textbooks, on the other hand, assume knowledge of measure theory and therefore discuss the Heston model in the language of measure theory ([8] [10] [13] [16] [18]). We aim to make the Heston model accessible to a general undergraduate audience by emphasizing intuitive notions in order to avoid any prerequisite knowledge of measure theory. Our paper is further motivated by the recent work of Mendoza, who investigates the Heston model and its past implementation in the Mexican Stock Exchange, where it was used to price index options on non-active trading days [15]. We interpret such work as indicative of the Heston model's empirical and practical applicability, as well as historical financial significance.

This paper begins by introducing relevant financial terminology and the Black-Scholes option pricing model. We then turn to the Heston model, explore its historical significance, its limitations, and the intuition surrounding the model. Thereafter, we discuss a modification to the Heston semi-analytic option pricing formula–one intended to correct a deficiency in the original formula referred to as *Little Heston's Trap* [1] (also see [19] for details). We discuss simulation methods relevant to the Heston model and offer a sample of our simulation code for curious readers.

2 Background: Finance

2.1 Financial Terminology

A nominal sum of money depreciates in real terms over time (for example, as a result of inflation). Therefore, borrowers of assets in the present make interest payments into the future which protect the lender against the possible future devaluation of the assets. In addition, the inflation-adjusted portion of an interest rate (called the real interest rate) serves to compensate for delayed consumption. Higher interest rates are typically indicative of a volatile economic climate or of substantial risk of default on the part of the borrower. As such, interest rates reflect a general perception of future real money value, a compensation for late consumption and the level of risk associated with particular financial transactions. In this framework, we can think of the *risk-free interest rate* as a rate which accounts for the general perception of future real money value as well as the compensation for late consumption with no risk of default or loss of funds. In finance, investors typically believe that U.S. Treasury bonds receive a risk-free rate. This stems from the belief that the U.S. is credible and will repay its debts; thus, all risk associated with the Treasury interest rate is neutralized.

With the risk-free interest rate in mind, we now introduce the concept of a financial derivative–a financial instrument whose payoff depends upon the future value of underlying assets. Among the different types of traded derivatives are forwards, futures, and options. Idiosyncratic in nature, they allow investors to customize holdings through various investment strategies. In this paper, we consider European call and put options which we abbreviate below as *call option* and *put option* for notational convenience. Later we also exclude the word *option*, choosing sometimes to simply write *call* or *put*. A focused exposition of these particular options best suits the purposes of our paper; we feel such an exposition offers more insight than would an overly general discussion of financial derivatives.

A *European call option* gives the option buyer the right to buy an underlying asset from the option seller at a future maturity date, T, for a predetermined future price (called the strike price), K, where the value of the asset at time t is denoted S_t with $0 \leq t \leq T$.

A *European put option* gives the option buyer the right to sell an underlying asset to the option seller at a future maturity date, T, for a predetermined future price (called the strike price), K, where the value of the asset at time t is denoted S_t with $0 \leq t \leq T$.

All financial derivatives involve two parties–a buyer and a seller. At the time of the contract, both an underlying asset (usually a stock, bond, foreign currency, etc.) and a future maturity date, T, are specified, while the future value of the asset S_T remains unknown. The negotiation of the strike price K results from a compromise reached by the parties, based on perceptions of the future asset value S_T. To clarify these concepts, we present brief examples of call and put options below.

Assume the underlying asset in question S_t represents the value of a stock which has an initial (present) stock price $S_0 = \$10$ and that the contract will mature in 1 year. Also assume that the buyer and seller agree upon the strike price $K = \$9$. Although the buyer of the option pays a price to buy the option from the seller at the initiation of the contract (which we will discuss later), our analysis here only considers the explicit payoff of the option at maturity date. So, when the contract matures after one year, consider the two cases where the stock price is either below or above the strike price. In this example, let $S_1 = 8$ and $S_1 = 10$, respectively.

Scenario 1: ($S_1 = 8$)

Call option contract: The buyer of the call option has the right to buy the stock at the strike price \$9 from the option seller at maturity. Since the buyer can buy the same stock on the market for \$8, a rational buyer will not exercise the option since the option is a right not an obligation for the buyer. Hence, both the buyer and the seller receive a payoff of \$0.

Put option contact: The buyer of the put option has the right to sell the stock for the strike price \$9 to the option seller. Since the buyer may purchase the same stock on the market for \$8 and sell it to the option seller for \$9, a rational buyer will exercise the option. Hence, the buyer receives a payoff of \$1 as a result of the put option, while the seller receives a payoff of $\$-1$. Formally, we can express the payoff of the buyer under these circumstances by $K - S_T$.

Scenario 2: $(S_1 = 10)$

Call option contract: The buyer of the call option has the right to buy the stock for \$9 from the option seller. Since the market price for the stock is \$10, the rational buyer will buy the stock from the seller and immediately sell the stock on the open market, thus netting a payoff of \$1, while the seller of the option realizes a payoff of $\$-1$. Formally, we can express the payoff of the buyer under these circumstances by $S_T - K$.

Put option contact: The buyer of the put option has the right to sell the stock at \$9 to the option seller. Since the buyer may sell the stock on the market for \$10, a rational buyer will not exercise the option. Both the buyer and seller receive a payoff of \$0.

Below, we mathematically summarize the buyer payoffs for both option types.

Call payoff for a buyer: $(S_T - K)^+$ where $(x)^+$ stands for $\text{Max}(x, 0)$.

Put payoff for a buyer: $(K - S_T)^+$ where $(x)^+$ stands for $\text{Max}(x, 0)$.

As the above example illustrates, rational buyers of call and put options decide whether or not to exercise options at maturity, meaning they always receive non-negative payoffs. To compensate the option seller for this right, option buyers pay a price for the purchased option–called a *premium*–at initiation of the contract which we mentioned earlier is not included when calculating an option's payoff. Understandably, the topic of arbitrage-free option pricing receives considerable attention in the financial world. An arbitrage opportunity is defined as taking advantage of a price difference in different markets to gain a risk-free profit. Arbitrage-free implies there is no arbitrage opportunity. Indeed most of the upcoming mathematics we consider resulted from investigations seeking to derive arbitrage-free call and put premiums.

By construction, buyers of call desire that the price of the underlying asset rises above the strike price so as to exercise the option and receive a positive payoff; the opposite holds for buyers of put. Within the context of an entire transaction, a null payoff is least desirable for the buyer of either option, since the buyer incurs the upfront loss of the premium. On the other hand, the null payoff is optimal from the option seller's perspective, since this implies the seller earns the premium without incurring financial loss.

For European call and put options with the same underlying asset, same maturity date, and same strike price (and assuming no dividends are paid out), there is a well-known simple algebraic relation, $P + S = C + Ke^{-rT}$, referred to as put-call parity. Here, C and

P represent the current call and put option prices (premiums) respectively, S represents the present underlying asset price, and K represents the strike price. In addition, r denotes the risk-free rate of interest whereas T denotes the maturity time (with a starting time of zero). By this relation, if we know the call price, it is easy to determine the put price and vice versa.

If we knew the future would yield $S_T > K$, the fair price of the call would be $e^{-rT}(S_T - K)$, the present value of the future payoff. Since no person has perfect knowledge of the future, it becomes necessary to model the behavior of S_t as a stochastic trajectory approaching maturity time T.

Derivatives serve as tools for efficiently and inexpensively managing risk exposure. They allow investors and portfolio managers to customize investment holdings and manage corresponding levels of risk. Trading derivatives offers all the convenience of trading assets without the burden of physically acquiring and selling assets. Largely for these reasons, the hedging and insurance functionality of derivatives has led to the widespread expansion of worldwide derivative markets.

The rest of our paper focuses on put options. There are several reasons for this, the foremost being that put options play a more significant role in the financial world than do call options. This stems from their use as a hedging tool against the devaluation of assets, since as we saw above, the payoff received from puts is positive when the asset value decreases below the strike price at maturity. When comparing possible future outcomes, it is intuitively more desirable to pay a small initial premium rather than risk a large percentage devaluation of one's investment. It is worth noting that by using put-call parity, the rest of this paper may be applied in a call option setting.

3 The Black-Scholes Model

3.1 History

Before discussing option pricing models, we must understand the behavior of S_t governing the option price. Moreover, it is crucial to explore the randomness involved when modeling a trajectory for S_t. For this we turn to the concept of Brownian motion. The ideas behind Brownian motion were applied to the field of economics by Louis Bachelier, who in his 1900 Ph.D. thesis, "The Theory of Speculation", presented a stochastic analysis of stock and option markets. Later, Albert Einstein applied Brownian motion to physics. In mathematics, Brownian motion has been revisited and is now often referred to as a *Wiener process*, named after Norbert Wiener. Robert C. Merton and Paul A. Samuelson modeled the behavior of financial markets with Brownian motion. Through similar work emerged the seminal Black-Scholes option pricing formula which, along with its fundamental assumption that stock price follows geometric Brownian motion, has left a lasting mark on the financial world.

A *Brownian motion* (Wiener Process) $W(t)$ is intuitively a position function of time t, beginning initially at 0, namely, $W(0) = 0$. Within a single time interval $[s, t]$, the difference $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$. For non-overlapping time intervals, the behavior of the function is independent, namely $W(v)-W(u)$ and $W(t)-W(s)$ are stochastically independent for the disjoint intervals $[u,v]$ and $[s,t]$. Interestingly, Brownian motion $W(t)$ is continuous but nowhere differentiable, nowhere monotonic, has uncountably many zeroes, and has unbounded variance. We can explore these first properties directly via discretization. For example, let $W(0)=0$ and sample $W(1)$ from a Gaussian distribution, $\mathcal{N}(0,1)$, and then connect $W(0)$ and $W(1)$ with a line segment. Similarly, we can model $W(2)-W(1)$ upon sampling from a Gaussian distribution $\mathcal{N}(0,1)$ and connecting $W(1)$ to $W(2)$. By continuing this process, we can generate a sample path (also called a trajectory) for $W(t)$ with the size of the time interval equal to one. Although this $W(t)$ is not itself an example of Brownian motion, when the size of the time interval approaches zero, the resulting limit process is Brownian motion. For example, we can build a Brownian motion path for the time interval $[0,t]$. Let $W(0)=0$ and pick the time interval $[0,t/n]$ where n is a positive integer. Then sample $W(t/n)$ from $\mathcal{N}(0,t/n)$, which is a normal distribution with mean 0 and variance t/n. Connect $W(0)$ and $W(t/n)$ by a line segment. Similarly, we can model $W(2t/n)-W(t/n)$ upon sampling from a Gaussian distribution $\mathcal{N}(0,t/n)$ and joining $W(t/n)$ to $(W2t/n)$ via a line segment. By continuing this process, we generate a trajectory for $W(t)$. As n tends to infinity, the resulting process is a Brownian motion.

Since we insist that assets have values strictly greater than zero, we must modify Brownian motion to account for this requirement and refer to the modification as geometric Brownian motion. The stochastic differential equation associated with geometric Brownian motion and the modeling of stock price evolution is of the form

$$dS_t = \mu S_t dt + \sigma S_t dW(t) \tag{1}$$

where μ and σ are annual drift (also called annual growth rate or expected return) and volatility constants, respectively. (Notice that the isolated component $dS_t = \mu S_t dt$ is a simple ordinary differential equation modeling exponential growth, where μ represents the annual growth rate.) As before, S_t represents the value of the underlying asset at a given time, t. The term $W(t)$ is a Wiener process–the stochastic element of the equation–while $dW(t)$ is the generalized derivative of the Wiener process and can be taken to be a sample from a Gaussian distribution with mean zero and variance dt. Note that a higher expected growth of stock corresponds to a larger value for μ. Similarly, the larger the value of σ, the greater the volatility of the stock trajectory; moreover, the longer the range of time t, the greater the volatility of S_t. Discretizing the above equation permits the trajectory of S_t to be simulated, which we will explore further in our section on simulation.

In the 1970s, Fischer Black and Myron Scholes of M.I.T. published ground-breaking research addressing the puzzling question of how to approach the pricing of certain options [4]. First, they assume that stock price obeys geometric Brownian motion (see Equation 1). Then, they use Ito's Lemma [12] to find the stochastic process of a financial derivative. They design a risk-free portfolio combining one financial derivative and fractional shares of underlying stock. Under the assumption of no arbitrage–the absence of risk when money receives a risk-free interest rate–they derive the Black-Scholes partial differential equation. By solving the Black-Scholes PDE with the boundary condition for call/put, they derived the famous Black-Scholes call/put option pricing formula [4].

In the Black-Scholes differential equation [12], the annual expected return μ does not influence option pricing. The intuition behind this phenomenon lies in the idea that the Brownian motion randomness in the financial derivative can be hedged away (i.e. mitigated) by the Brownian motion randomness in the underlying stock. Along with this assumption, it is assumed that the investing climate is *risk-neutral*, that is, stock grows at the risk-free rate in which money should be discounted back by the risk-free rate. By constructing a partial differential equation model in such a way that accounts for continuously hedging risk, their model implies the existence of a single fair option price. In the midst of this construction, they demonstrate the use of processes which sufficiently consider an arbitrage-free pricing environment, which we heuristically claimed earlier is necessary. Other assumptions made by the Black-Scholes model include the absence of transaction costs, taxes, and dividends, a continuous trading environment, a constant rate of risk-free interest at which investors are able to borrow, the permissibility of short-selling assets, and the divisibility of securities [4].

3.2 Black-Scholes Model

A rigorous derivation of the Black-Scholes option pricing formula demands an in-depth knowledge of finance. A sketch of the proof can be found in Hull [12].

Theorem 3.1. *Black-Scholes Formula: An option's fair price is the discounted expected value of that option's payoff within a risk-neutral world. In particular, the fair price for a put option P is given below, where r is the risk-free rate and $\mathcal{N}(.)$ stands for the cumulative distribution function of standard normal distribution.*

$$P = e^{-rT}\mathbb{E}\left[(K - S_T)^+\right] = Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$
$$d_1 = \frac{Log\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{Log\left(\frac{S_0}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Both $\mathcal{N}(-d_2)$ and $\mathcal{N}(-d_1)$ have important practical significance. The term $\mathcal{N}(-d_2)$ represents the risk-neutral probability that S_T is less than K–and is therefore the probability that the put option is exercised in the risk-neutral world. Elementary calculus allows us to calculate the put price sensitivity with respect to the underlying stock, namely, $\frac{\partial P}{\partial S_0}$, which is equal to $-\mathcal{N}(-d_1)$. Here we can see how mathematical intuition applies to finance; in particular, the partial derivative implies that if stock price goes up by $1, the put price will go down by approximately $\mathcal{N}(-d_1)$ dollars. So, we can construct a risk-free portfolio by buying one put and buying $\mathcal{N}(-d_1)$ shares of stock. This is risk-free because the portfolio value does not depend upon the movement of the stock price. And yet, notice that both S_t and $\mathcal{N}(-d_1)$ are functions of time with the implication that the risk-free portfolio adjusts instantaneously. (In finance, the term $\mathcal{N}(-d_1)$ is referred to as the delta hedge ratio and is used extensively by investment banks and firms.)

4 The Heston Model

4.1 The Model

The Black-Scholes model assumes that volatility, σ, is constant over time. In reality, this assumption is problematic. When considering real financial data, such as stock indices, one observes volatility clustering periods in which volatility is noticeably high or low, followed by periods with opposite behavior. Moreover, in a real market, when stock prices decrease the volatility tends to be heightened. Heston's model accounts for such behavior by treating σ as variable and by incorporating a correlation term.

Another issue not addressed by Black and Scholes is that of *volatility smile*: if volatility is constant for a given stock index, numerous different options each having different strike prices should have the same assumed volatility. If we regard the market-observed option prices of different strikes as inputs, using the Black-Scholes formula we can solve for the unknown market volatility, called the implied volatility. In theory, implied volatility should be constant for different options with different strike prices that have the same underlying asset. However, if we graph strike price as the independent variable and and implied volatility as the dependent variable in $\mathbb{R}^2$, the visual resembles a *smiling* curve, rather than a horizontal line. Heston's model accounts for the phenomenon of volatility smile.

The Heston model assumes volatility follows a random process negatively correlated with the evolution of the asset price as indicated by market data. This has been regarded as a valuable improvement over the Black-Scholes model and demonstrates the volatility smile phenomenon [11]. The value of this improvement is evidenced further in the application of Heston's model to the Mexican financial market [15]. It is important to note, however, that the Heston model remains far from perfect; most noticeably, its framework cannot adequately explain stock market crashes which result from large jumps in the magnitude of financial returns. The more-recent Bates model [2] attempts to improve upon the Heston model by addressing this issue; however, a deeper exploration of the Bates model is beyond the scope of this paper.

The Heston model [11] extends the Black-Scholes model by modeling volatility, denoted $\sqrt{V(t)}$, as a stochastic process. Below are the stochastic differential equations that characterize the Heston model in a risk-neutral world.

$$
\begin{aligned}
&dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dW^S(t) \\
&dV(t) = \kappa\left(\theta - V(t)\right)dt + \sigma_V\sqrt{V(t)}dW^V(t) \\
&dW^S(t)dW^V(t) = \rho dt
\end{aligned}
$$

where r is the risk-free rate, $V(t)$ is the variance which is the volatility squared, θ is the long run mean to which the $V(t)$ process reverts, κ is the mean reversion coefficient, σ_V is the volatility of the variance process, $W^S(t)$ is the Brownian motion for the stock process, $W^V(t)$ is the Brownian motion for the variance process $V(t)$ and the two Brownian motions have a correlation of ρ which is usually negative.

Examining the equation for stock variance, $V(t)$, we see that:

$$dV(t) = \underbrace{\kappa\left(\theta - V(t)\right)dt}_{\text{deterministic component}} + \underbrace{\sigma_V\sqrt{V(t)}dW^V(t)}_{\text{Brownian motion component}}$$

Notice that the deterministic component resembles a mean-reversion ordinary differential equation with mean θ and reversion coefficient κ. This dictates the long run behavior of variance, and in this process, variance approaches the long run mean. The idea of mean reversion is central to economics, whereupon disruption from equilibrium results in temporarily high volatility before falling and converging to the lower, long run mean of volatility. Similarly in finance, interest rate processes are frequently modeled using mean reversion.

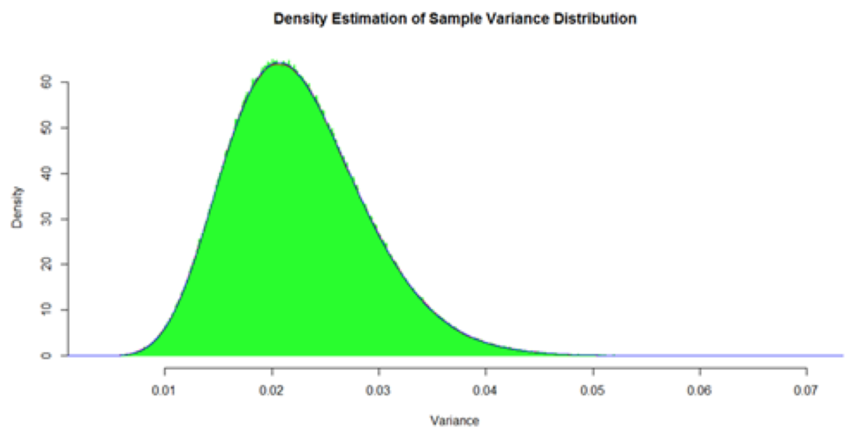
Now we would like to explore the probability density of the variance $V(t)$ in the Heston model. To do so, we turn to a paper by Broadie et al. [5], in which the Heston model is considered within the context of monthly ($T = 1/12$) puts for S&P 500 index futures [12]. (See Hull for an explanation of S&P index futures. For intuitive purpose, we regard the index futures as similar to the stock index.) In their paper, Broadie et al. aim to show that over-priced put options in the market are consistent with various option pricing models including those of Heston and Bates. While we will not discuss the details of their research, we will borrow their Heston model parameters ($\kappa = 5.33$, $\theta = 0.0225$, $\sigma_V = 0.14$) which are calibrated on 215 months of S&P index futures market data from August 1987 to June 2005. When compared to a constant variance of 0.0225 (volatility of 0.15) in the Black-Scholes model, this distribution of variance resembles the market variance in the period between 1987 and 2005.

We will use an Euler scheme to generate the distribution of the sample variance. It can be shown that that the implemented Euler scheme converges to the variance process as the time interval length tends to zero. Note that the following equation never allows V_t to become negative, a necessary constraint given the square-root operator on V_t:

$$V_{t+\Delta t} = [(V_t + \kappa(\theta - V_t)\Delta t + \sigma_V\sqrt{(V_t)}(\Delta W)_t)]^+.$$

We generate 3,000,000 data points of V_t to estimate the distribution for the variance. The steps are as follows: We choose $V_0 = 0.0225$ as the starting point of the Gibbs sampler. Using the equation above, we generate a sequence of 30,010,000,000 pairs of $\{V_t, V_{t+\Delta t}\}$ where $\Delta t = (1/12)/10,000$ to reduce discretization bias. We also discard the first 10,000,000 pairs of $\{V_t, V_{t+\Delta t}\}$ to avoid starting point bias and then take a subsample of every 10,000th $\{V_t, V_{t+\Delta t}\}$ to minimize the effects of correlation. This yields a sample of 3,000,000 pairs of $\{V_t, V_{t+\Delta t}\}$ from which we obtain an estimate of the marginal distribution of V_t.

The following graph is the density of variance by the above simulation. Notice that the sample variance distribution is far from constant for real market data.



4.2 Pricing Options Using the Heston Model

Based on Heston's model, the fair price for a put option is $P = Ke^{-rT}(1 - F_2) - S_0(1 - F_1)$. When compared to the Black-Scholes model, $1 - F_2$ is the probability of S_T assuming a value less than K in a risk-neutral world. The term $-(1 - F_1)$ is the delta hedge ratio which itself is a probability in a different measure. Since our paper is targeted for an undergraduate audience, we exclude further discussion of measures and the details explaining why the delta hedge ratio is a probability in a different measure.

In order to calculate the above probabilities F_1 and F_2, we need to define the characteristic function of a random variable. Letting X be a random variable, we can define the characteristic function $f(\phi) = E(e^{iX\phi})$, where E is the expected value operator and $i = \sqrt{-1}$. In this way, the characteristic function is the Fourier transformation of the probability density function of X. Conversely, upon knowing the characteristic function, we can determine the probability density and cumulative distribution functions of the random variable X via inverse Fourier transformations. In general, by evaluating the cumulative distribution function at the point K (or in our case at $\ln(K)$), we obtain the probability F_1 or F_2.

Now we offer the modified version of Heston's semi-analytic option pricing formula. Our presentation modifies that in [19] for notational clarity. Similar to Zhu, we express the probabilities F_1 and F_2 in terms of the inverse Fourier transformation of *characteristic function*

$f_j(\phi)$ where $j = 1, 2$.

$$
\begin{aligned}
F_j &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(f_j(\phi)\frac{e^{-i\phi\ln K}}{i\phi}d\phi\right)\\
f_1(\phi) &= E^{Q_1}\left[e^{i\phi\ln S(T)}\right]\\
f_2(\phi) &= E^{Q_2}\left[e^{i\phi\ln S(T)}\right]
\end{aligned}
$$

The derivation of $f_1(\phi)$ and $f_2(\phi)$ is omitted here because it involves measure theory, but they can be expressed as

$$f_j(\phi) = e^{D_j(\phi)V_0 + C_j(\phi) + i\phi\ln S_0}, \text{ where} \tag{2}$$

$$
\begin{aligned}
D_j(\phi) &= \frac{b_j - d_j - \rho\sigma i\phi}{\sigma^2}\frac{1 - e^{-d_jT}}{1 - q_je^{-d_jT}}\\
C_j(\phi) &= i\phi rT + \frac{\kappa\theta}{\sigma^2}\left[\left(b_j - d_j - \rho\sigma i\phi\right)T - 2\ln\left(\frac{1 - q_je^{-d_jT}}{1 - q_j}\right)\right]\\
g_j &= \frac{b_j + d_j - \rho\sigma i\phi}{b_j - d_j - \rho\sigma i\phi}\\
b_1 &= \kappa - \rho\sigma\\
b_2 &= \kappa\\
d_1 &= \sqrt{\left(\rho\sigma i\phi - b_j\right)^2 + \sigma^2\left(\phi^2 - i\phi\right)}\\
d_2 &= \sqrt{\left(\rho\sigma i\phi - b_j\right)^2 + \sigma^2\left(\phi^2 + i\phi\right)}\\
q_j &= \frac{1}{g_j}
\end{aligned}
$$

We note that a typo found in Zhu's book is corrected above.

4.3 On Little Heston's Trap

Both the original formula for put price in Heston's 1993 paper and the modified version of the semi-analytic option pricing formula mentioned in the previous section require the evaluation of a complex logarithm. Since branch-cutting may occur, complex logarithms can be problematic. As mentioned previously, Albrecher discovered and offered a remedy for the branch-cutting error in Heston's paper [1]. In this section, we discuss branch-cutting problems as they pertain to the Heston model.

Note that the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$ has a unique output for each input, i.e, $f(z) = z^2$ is well-defined for each $z \in \mathbb{C}$. Now consider the inverse function $g(z) = \sqrt{z}$. This function does not exhibit this same property. One might ask, *What is* $g(-4)$*?* Both $f(-2i)$ and $f(2i)$ evaluate to -4. However, for g to be a well-defined function, it must have a single output for each input. As it stands, for g we are left to select one of two possible values as the output. This issue is commonly referred to as a branch-cutting problem and also affects complex logarithms; the term different branch refers to different output values.

Many software packages automatically select the principal branch when calculating complex logarithms, which can create discontinuities. Precisely this happened when automated algorithms were used for these computations after the initial publication of the Heston's semi-analytic option price formula. As a result, researchers and financial analysts miscalculated put prices, all-the-while unaware that their algorithms were cutting the wrong branches, so to speak.

The modified formula in Albrecher's paper [1] does not exhibit branch-cutting discontinuities in practical application [19]. Using the code provided in the appendix, we can compare results generated from the original and modified formulas. The first table (below) indicates the chosen parameter values.

$S_0 = 100$	$T = 6$	$r = 0.04$	$v_0 = 0.0225$
$\theta = 0.04$	$\sigma = 0.3$	$\rho = -0.5$	$\kappa = 2$

The following table compares fair put prices determined by Heston's original semi-analytic option pricing formula to those determined by the modified version.

$Strike K$	Put Price (original formula)	Put Price (modified formula)
70	2.49029	2.2157
80	5.05087	3.7305
100	10.8598	8.41705
120	17.5012	15.4256
130	21.2458	19.7636

These two methods produce noticeably different results! As such, improper branch-cutting choices can produce incorrect numerical solutions with severe economic consequences. Depending on the specified financial contract, mispriced options can lead to hugely skewed financial losses or gains.

5 Simulation

5.1 Determining Put Price by Simulating Stock Paths in the Black-Scholes Model

In order to get a derivative price today, we can simulate different stock price trajectories. Once final stock prices at maturity are simulated for different trajectories, we can calculate the put payoffs for different trajectories at maturity. Taking the arithmetic average of the payoffs for different trajectories at maturity and discounting back by the risk-free rate allows us to estimate today's model implied put price. As such, we focus on simulating trajectories here, keeping in mind that simulation of these models is also useful for pricing complicated financial derivatives other than options in which analytic approaches may not exist.

A common way to simulate stock trajectories involves the discretization of stochastic differential equations, as was briefly mentioned earlier. Recall that for the Black-Scholes model in a risk neutral world:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t),$$

where $dW(t) \sim \mathcal{N}(0, dt)$.

To discretize this equation, we replace dS, dW, and dt with $\Delta S, \Delta W$, and Δt, letting $\Delta W \sim \mathcal{N}(0, \Delta t)$. Ideally, Δt is set to be as small as computationally feasible.

If we apply Ito's Lemma, we can rewrite the Black-Scholes model in a risk neutral world as

$$d\,(\ln S) = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW,$$

in turn producing

$$\Delta\,(\ln S) = \left(r - \frac{1}{2}\sigma^2\right) \Delta t + \sigma \Delta W.$$

We can use this equation with large time steps without discretization bias if we are only interested in the final stock price S_T. That is, consecutive steps in a simulation can be summed up and expressed precisely with a single equation, in which case we have a global formula. If we sum n steps of equal size Δt, we get

$$\ln S_n - \ln S_1 = n\left(r - \frac{1}{2}\sigma^2\right) \Delta t + \sigma \Delta W_n,$$

where $\Delta W_n \sim \mathcal{N}(0, n\Delta t)$. Note that $S_n = S_T$ and $n\Delta t = T$.

Since sampling from a Gaussian distribution can be computationally expensive, this algorithm presents an efficient method for simulating final stock prices, a prerequisite for calculating the price of a put option.

5.2 Determining Put Price by Simulating Stock Paths in the Heston Model

Given non-constant volatility, there is no global algorithm that efficiently extends to large time steps without introducing discretization bias. We cannot therefore use Ito's Lemma formula to get a globalized formula for the final stock price. To achieve reasonable accuracy with the Heston model, one must discretize the stock price process with small time intervals such as to limit the discretization error.

Instead of using the Euler Scheme which we discussed previously, Zhu [20] presents a more efficient algorithm for simulating the volatility process (the square root of variance). Zhu uses Ito's Lemma to convert Heston's variance process V_t to a volatility process denoted v_t. Zhu then discretizes the volatility process v_t. The formulas for this, referred to as the *transformed volatility scheme*, are given below. See the Appendix for a sample program that performs this algorithm.

$$\begin{aligned} S_{t+\Delta t} &= S_t e^{\left(r-\frac{1}{2}v_t^2\right)\Delta t + v_t \Delta W_t^S} \\ v_{t+\Delta t} &= v_t + \frac{1}{2}\kappa\Delta t\left(\frac{\beta - v_t e^{-\frac{1}{2}\kappa\Delta t}}{1-e^{-\frac{1}{2}\kappa\Delta t}} - v_t\right) + \frac{1}{2}\sigma_v \Delta W_t^V \\ \beta &= \sqrt{\left(\theta + \left(v_t^2-\theta\right)e^{-\kappa\Delta t} - \frac{\sigma_V^2}{4\kappa}\left(1-e^{-\kappa\Delta t}\right)\right)^+} \end{aligned}$$

Note we must keep Δt small even though we are interested only in the final stock price S_T. This follows from the need to generate an entire sample path over our specified time interval–in which volatility is highly variable. Finally, taking the arithmetic average of the payoffs of the put for different trajectories and discounting back by the risk-free rate allows us to estimate the present model-implied put price.

6 Conclusion

We have introduced topics relating to option pricing models which are typically unavailable to undergraduate students. We discussed how *branching cutting* relates to the world of finance, described related simulation techniques, and provided machinery for option pricing. Interested undergraduate readers can use the tools and ideas we described to further their own studies and research related to option pricing models. It is our hope that this work may contribute towards bridging an existing gap in the undergraduate financial mathematics literature.

7 Appendix

7.1 R Code: Estimating Put Price Using Simulation of the Heston Model

Recall that fair put price is the discounted (by risk-free rate) expected value of the put payoff in a risk-neutral world (with a risk-free growth rate). In this code, we estimate the price of call and the price of put by averaging the discounted expected payoffs via simulation using the Heston model.

```
library(MASS)

rm(list=ls(all=TRUE))

kappa = 2
theta = 0.04
rho = -0.5
sigma = 0.3
rf = 0.04
T = 6
s0 = 100
v0 = 0.0225
K = 70
trajectories = 800000
step = 200
dt = T/step
cdt = kappa*dt
hcdt = kappa*dt*0.5
ctdt = cdt*theta
rtdt = dt^0.5
cons0 = exp(-hcdt)
cons1 = 1-cons0
cons2 = exp(-cdt)
cons3 = 1-cons2
cons4 = 4*kappa
cons5 = 0.5*sigma*rtdt
cons6 = sigma^2/cons4*cons3
mat = matrix(c(1, rho, rho, 1),nrow=2, ncol=2)

Call = rep(NA, trajectories)
Put = rep(NA, trajectories)
```

```
for(j in 1:trajectories){
s = s0
vol = v0^0.5
z = mvrnorm(step,c(0, 0), mat)
for(i in 1:step){
s = s*exp((rf-1/2*vol^2)*dt+vol*rtdt*z[i,1])
vol = vol+hcdt*((max(theta+(vol^2 - theta)*cons2-cons6, 0)^0.5-vol*cons0)/cons1-vol)+cons5*z[i,2]
}
Call[j] = exp(-rf*T)*max(s-K,0)
Put[j] = exp(-rf*T)*max(K-s,0)
}
mean(Call)
mean(Put)
```

7.2 Mathematica Code: Estimating Put Price Using Modified Heston's Semi-Analytic Option Pricing Formula

$\kappa = 2;$ (*Mean Reversion Constant*)
$\theta = 0.04;$ (*The long run variance *)
$\rho = -0.5;$ (*Correlation*)
$\sigma_V = 0.3;$ (*Volatility of volatility*)
$r = 0.04;$ (*Risk − free rate*)
$T = 6;$ (*Contract length (in years)*)
$s_0 = 100;$ (*Initial stock value*)
$v_0 = 0.0225;$ (*Initial variance*)
$K = 70;$ (*Strike price*)

(*Internal Variables*)
$\mu_1 = 0.5;$
$\mu_2 = -0.5;$
$a = \kappa * \theta;$
$b_1 = \kappa - \rho * \sigma_V;$
$b_2 = \kappa;$
$k = K / s_0;$

$\text{d1[phi\_]:=}\sqrt{(\rho * \sigma_V * \text{phi} * I - b_1)^2 - \sigma_V{}^2 * \left(2 * \mu_1 * \text{phi} * I - \text{phi}^2\right)}$
$\text{d2[phi\_]:=}\sqrt{(\rho * \sigma_V * \text{phi} * I - b_2)^2 - \sigma_V{}^2 * \left(2 * \mu_2 * \text{phi} * I - \text{phi}^2\right)}$
$\text{q1[phi\_]:=}\frac{b_1 - \rho * \sigma_V * \text{phi} * I - \text{d1[phi]}}{b_1 - \rho * \sigma_V * \text{phi} * I + \text{d1[phi]}}$

$\mathbf{q2[phi\_]:=}\frac{b_2-\rho*\sigma_V*\mathrm{phi}*I-\mathrm{d2[phi]}}{b_2-\rho*\sigma_V*\mathrm{phi}*I+\mathrm{d2[phi]}}$

$\mathbf{D1[phi\_]:=}\frac{b_1-\rho*\sigma_V*\mathrm{phi}*I-\mathrm{d1[phi]}}{\sigma_V{}^2}*\left(\frac{1-\mathrm{Exp}[-\mathrm{d1[phi]}*T]}{1-\mathrm{q1[phi]}*\mathrm{Exp}[-\mathrm{d1[phi]}*T]}\right)$

$\mathbf{D2[phi\_]:=}\frac{b_2-\rho*\sigma_V*\mathrm{phi}*I-\mathrm{d2[phi]}}{\sigma_V{}^2}*\left(\frac{1-\mathrm{Exp}[-\mathrm{d2[phi]}*T]}{1-\mathrm{q2[phi]}*\mathrm{Exp}[-\mathrm{d2[phi]}*T]}\right)$

$\mathbf{C1[phi\_]:=}r\mathrm{phi}IT+\frac{a}{\sigma_V{}^2}\left(\left(b_1-\rho*\sigma_V*\mathrm{phi}*I-\mathrm{d1[phi]}\right)T-2*\mathrm{Log}\left[\frac{1-\mathrm{q1[phi]}*\mathrm{Exp}[-\mathrm{d1[phi]}*T]}{1-\mathrm{q1[phi]}}\right]\right)$

$\mathbf{C2[phi\_]:=}r\mathrm{phi}IT+\frac{a}{\sigma_V{}^2}*\left(\left(b_2-\rho*\sigma_V*\mathrm{phi}*I-\mathrm{d2[phi]}\right)T-2*\mathrm{Log}\left[\frac{1-\mathrm{q2[phi]}*\mathrm{Exp}[-\mathrm{d2[phi]}*T]}{1-\mathrm{q2[phi]}}\right]\right)$

$\mathbf{Re1[phi\_]:=Re}\left[\frac{\mathrm{Exp}[\mathrm{C1[phi]}+\mathrm{D1[phi]}*v_0+I*\mathrm{phi}*\mathrm{Log}[1/k]]}{I*\mathrm{phi}}\right]$

$\mathbf{Re2[phi\_]:=Re}\left[\frac{\mathrm{Exp}[\mathrm{C2[phi]}+\mathrm{D2[phi]}*v_0+I*\mathrm{phi}*\mathrm{Log}[1/k]]}{I*\mathrm{phi}}\right]$

$P_1=\frac{1}{2}+\frac{1}{\pi}*\mathrm{NIntegrate}[\mathrm{Re1[phi]},\{\mathrm{phi},0,\mathrm{Infinity}\}];$

$P_2=\frac{1}{2}+\frac{1}{\pi}*\mathrm{NIntegrate}[\mathrm{Re2[phi]},\{\mathrm{phi},0,\mathrm{Infinity}\}];$

$\mathrm{PutPrice}=K*\mathrm{Exp}[-r*T]*(1-P_2)-\ s_0*(1-P_1)$ (*Put Price*)

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