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On the Center of Non Relativistic Lie Algebras

Tyler Gorshing

Abstract. The center of the Schrödinger Lie algebra is the Lie subalgebra generated by its center of mass. An explicit mathematical proof of this statement doesn't seem to be available in literature. In this paper, we use elementary matrix multiplication to prove it. We also investigate the case of the Galilei Lie algebra, the Harmonic Oscillator Lie algebra and the Heinsenberg-Weyl Lie algebra. We show by calculation that these non-relativistic Lie algebras have no center unless centrally extended.

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1 Introduction

The theory of Lie algebras is one of the most important subjects in both pure mathematics and mathematical physics. Among the existing Lie algebras, non-relativistic Lie algebras have attracted a lot of attention for many years because of their various applications (see [9]) in mathematical physics. In particular, a tremendous amount of papers have been written on the Schrödinger Lie algebra, the Galilei Lie algebra and the Harmonic Oscillator Lie algebra (see, for example, [1, 2, 5]).

It is legitimate and not unusual for a beginning physicist to wonder if the center of mass coincides with the center of gravity. In the same order of idea, a beginning Lie algebra learner wonders if the center of mass of a centrally extended non-relativistic Lie algebra coincides with its center as a Lie subalgebra, since the concept of center of mass becomes very complex as we move from classical mechanics to non-relativistic quantum physics [8]. In this paper, we use the techniques performed by G. Biyogmam in [2] to calculate the center of the non-relativistic Lie algebras mentioned above. We provide all the details for the Schrödinger algebra case and state the results for the others as the calculation is similar. As a result, we show that when the Schrödinger, Galilei, Harmonic Oscillator, and Heinsenberg-Weyl Lie algebras are massless, these Lie algebras have no center; however, when they are centrally extended with a mass M, our calculations show that the center of the Lie algebra is the subalgebra generated by M.

This paper is structured as follows: In Section 2, we provide some preliminaries. Section 3 is divided into three subsections. In Section 3.1, we provide a convenient basis of the Schrödinger Lie algebra along with the calculation of its center. In Section 3.2 we briefly outline the proof for the case of the Galilei Lie Algebra. The cases of the Harmonic Oscillator Lie Algebra and the Heisenberg-Weyl Lie Algebra are respectively studied in Section 3.3 and Section 3.4.

2 Preliminaries

Let us recall a few definitions:

Definition 2.1. [6] A *Lie Algebra* over the field $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ with a $\mathbb{K}$-bilinear map $[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ which is skew-symmetric, meaning

$$[x,y]=-[y,x] \quad \text{for all } x,y\in\mathfrak{g},$$

and satisfies the Jacobi identity, which is

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

Definition 2.2. [6] A subspace $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $\mathfrak{s}$ has the structure of a Lie algebra when endowed with the restriction of the bilinear operation of $\mathfrak{g}$ on $\mathfrak{s}\times\mathfrak{s}$.

Definition 2.3. [7] Let $\mathfrak{g}$ be a Lie algebra. The *center* $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is the subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, g] = 0 \text{ for all } g \in \mathfrak{g}\}.$$

In the following remark, we provide a constructive proof of a well-known result in the field of linear algebra.

Remark 2.4. Let V be a finite dimensional vector space, V^* the vector space of all linear transformations defined from V to $\mathbb{R}$ and $M_{n\times n}(V)$ the vector space of all $n \times n$ matrices with coefficients in V, then there is a vector space isomorphism

$$V \times V^* \cong M_{n\times n}(V).$$

Indeed, assume that $\{x_1, x_2, ..., x_n\}$ is a basis of V, and consider the linear maps $\frac{\partial}{\partial x_j} : V \to \mathbb{R}$ defined by

$$\frac{\partial}{\partial x_i}(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Then clearly $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}$ is a basis of V^* and the n^2 vectors $\{x_i \frac{\partial}{\partial x_j}\}_{i,j=1}^{n}$ constitute a basis of $V \times V^*$. Now let $\{e_{ij}\}_{i,j=1}^{n}$ be a basis of $M_{n\times n}(V)$, where e_{ij} is a $n \times n$ matrix where 1 is in the i^{th} row, j^{th} column and 0 everywhere else. Then it is clear that the map $\alpha : V \times V^* \to M_{n\times n}(V)$ defined by $\alpha(x_i \frac{\partial}{\partial x_j}) = e_{ij}$, is bijective and linear.

As a consequence of the above remark, we have the following:

Corollary 2.5. The vector space V^* operates on $V \times V^*$ via multiplication of matrices and for all $k \in \{1, 2, \ldots, n\}$ the maps

$$\frac{\partial}{\partial x_k}\big(x_i \frac{\partial}{\partial x_j}\big) = \begin{cases} \frac{\partial}{\partial x_j}, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$

are linear.

Proof. It is easy to check that if e_k is a column matrix where 1 is in the k^{th} position and zero elsewhere, then multiplying these matrices yields to

$$e_k\big(e_{ij}\big) = \begin{cases} e_j, & \text{if } i = k \\ 0, & \text{if } i \neq k. \end{cases}$$

We conclude using the isomorphism α above. □

Remark 2.6. When endowed with the bracket operation
$[-, -] : M_{n\times n}(V) \times M_{n\times n}(V) \to M_{n\times n}(V)$ defined by $[a, b] = ab - ba$, the vector space $M_{n\times n}(V)$ becomes a Lie algebra. It is called general linear Lie algebra and denoted $\mathfrak{gl}(n)$.

3 The Schrödinger Lie Algebra

In this section, we provide a basis of the Schrödinger Lie Algebra and a few of its Lie subalgebras in term of the vectors $\{x_i\frac{\partial}{\partial x_j}\}_{i,j=1}^{n+2}$ and calculate their centers.

Denote by $\mathfrak{sch}_n$, the Schrödinger Lie algebra. Using the isomorphism of remark 2.4 and the matrix representation of a typical element of $\mathfrak{sch}_n$, we provide below a basis of the Schrödinger Lie algebra. Assume that $\mathbb{R}^n$ is given the coordinates $(x_1, x_2, ..., x_n)$, then the set

$$\Big\{x_{ij}, \alpha_n, \beta_n, \gamma_n, x_i\frac{\partial}{\partial x_{n+1}},\ x_i\frac{\partial}{\partial x_{n+2}};\ \ 1 \le i < j \le n\Big\}$$

is a basis of the Schrödinger algebra $\mathfrak{sch}(n)$, where

$x_{ij} := -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i}, \quad 1 \le i < j \le n$ (Rotations),

$\alpha_n := -x_{n+1}\frac{\partial}{\partial x_{n+1}} + x_{n+2}\frac{\partial}{\partial x_{n+2}}$ (Dilation),

$\beta_n := x_{n+1}\frac{\partial}{\partial x_{n+2}}$ (Time translation),

$\gamma_n := -x_{n+2}\frac{\partial}{\partial x_{n+1}}$ (Conformal transformation),

$x_i\frac{\partial}{\partial x_{n+1}} \quad 1 \le i \le n$ (Galilean boosts),

$x_i\frac{\partial}{\partial x_{n+2}} \quad 1 \le i \le n$ (Space translations),

and the non zero brackets are:

$$[x_{ij}, x_{ik}] = x_{jk}, \quad [\alpha_n, \beta_n] = -2\beta_n, \quad [\alpha_n, \gamma_n] = 2\gamma_n, \quad [\beta_n, \gamma_n] = \alpha_n,$$

$$[x_{ij}, x_i\frac{\partial}{\partial x_{n+1}}] = x_j\frac{\partial}{\partial x_{n+1}}, \quad [x_{ij}, x_i\frac{\partial}{\partial x_{n+2}}] = x_j\frac{\partial}{\partial x_{n+2}}, \quad [\alpha_n, x_i\frac{\partial}{\partial x_{n+1}}] = x_i\frac{\partial}{\partial x_{n+1}},$$

$$[\alpha_n, x_i\frac{\partial}{\partial x_{n+2}}] = -x_i\frac{\partial}{\partial x_{n+2}}, \quad [\beta_n, x_i\frac{\partial}{\partial x_{n+1}}] = -x_i\frac{\partial}{\partial x_{n+2}}, \quad [\gamma_n, x_i\frac{\partial}{\partial x_{n+2}}] = x_i\frac{\partial}{\partial x_{n+1}}.$$

These brackets are calculated using matrix multiplication and the bracket of $\mathfrak{gl}(n)$. For example,

$$\begin{aligned}[x_{12}, x_{13}] &= x_{12}x_{13} - x_{13}x_{12}\\ &= (-x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1})(-x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_1}) - (-x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_1})(-x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1})\\ &= -x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_2}\\ &= x_{23}.\end{aligned}$$

Remark 3.1. The Schrödinger Lie algebra is said to be centrally extended when it is provided an additional basis element M called mass, and satisfying $[M, X] = 0$ for all $X \in \mathfrak{sch}_n$.

In this case the interaction between the Galilean boosts and the space translations is not always trivial. More precisely [5], $[x_i\frac{\partial}{\partial x_{n+1}}, x_j\frac{\partial}{\partial x_{n+2}}] = \delta_{ij}M$ for all $i, j = 1, 2, \ldots, n$, where δ_{ij} =1 if $i = j$ and 0 if $i \neq j$.

3.1 The center of the Schrödinger Lie Algebra

The following provides the center of the Schrödinger Lie algebra.

Proposition 3.2. The center of the centrally extended Schrödinger Lie algebra $\mathfrak{sch}_n$ with mass M coincides with the subalgebra generated by its mass; that is

$$\mathfrak{z}(\mathfrak{sch}_n) = \langle M \rangle \quad \text{for all} \quad n \geq 1.$$

Proof. We proceed by induction on n starting from $n = 2$ for clarity, since $\mathfrak{sch}_1$ is the only case where there are no rotation generators. The calculation of $\mathfrak{z}(\mathfrak{sch}_1)$ is analogous and easier than $\mathfrak{z}(\mathfrak{sch}_2)$. The following is a basis for $\mathfrak{sch}_2$:

$$B_{\mathfrak{sch}_2} = \{M, x_{12}, \alpha_2, \beta_2, \gamma_2, x_i\frac{\partial}{\partial x_3}, x_i\frac{\partial}{\partial x_4},\ i = 1, 2\}.$$

So for $X \in \mathfrak{z}(\mathfrak{sch}_2)$, we have

$$X = ax_{12} + bx_1\frac{\partial}{\partial x_3} + cx_2\frac{\partial}{\partial x_3} + dx_1\frac{\partial}{\partial x_4} + ex_2\frac{\partial}{\partial x_4} + f\alpha_2 + g\beta_2 + h\gamma_2 + iM$$

for some $a, b, c, d, e, f, g, h, i \in \mathbb{R}$. We then have

$$0 = [X, x_{12}] = a[x_{12}, x_{12}] + b[x_1\frac{\partial}{\partial x_3}, x_{12}] + c[x_2\frac{\partial}{\partial x_3}, x_{12}] + d[x_1\frac{\partial}{\partial x_4}, x_{12}] + e[x_2\frac{\partial}{\partial x_4}, x_{12}]$$
$$+ f[\alpha_2, x_{12}] + g[\beta_2, x_{12}] + h[\gamma_2, x_{12}] + i[M, x_{12}]$$

which gives $0 = b(-x_2\frac{\partial}{\partial x_3}) + c(x_1\frac{\partial}{\partial x_3}) + d(-x_2\frac{\partial}{\partial x_4}) + e(x_1\frac{\partial}{\partial x_4})$, and thus $b, c, d, e = 0$ by linear independence. So we are reduced to

$$X = ax_{12} + f\alpha_2 + g\beta_2 + h\gamma_2 + iM.$$

Similarly,

$$0 = [X, \alpha_2] = a[x_{12}, \alpha_2] + f[\alpha_2, \alpha_2] + g[\beta_2, \alpha_2] + h[\gamma_2, \alpha_2] + i[M, \alpha_2].$$

So,

$$[X, \alpha_2] = 2g\beta_2 - 2h\gamma_2 = 0$$

giving $g, h = 0$ and thus $X = ax_{12} + f\alpha_2 + iM$.
Also,

$$0 = [X, \beta_2] = a[x_{12}, \beta_2] + f[\alpha_2, \beta_2] + i[M, \beta_2] = -2f\beta_2$$

giving $f = 0$ by linear independence, and thus $X = ax_{12} + iM$.
Finally,

$$0 = [X, x_1 \frac{\partial}{\partial x_3}] = a[x_{12}, x_1 \frac{\partial}{\partial x_3}] + i[M, x_1 \frac{\partial}{\partial x_3}] = ax_2 \frac{\partial}{\partial x_3}$$

giving $a = 0$ by linear independence, hence $X = iM$ and thus $\mathfrak{z}(\mathfrak{sch}_2) \subseteq \langle M \rangle$. Since by definition $[M, X] = 0$ for all $X \in \mathfrak{sch}_2$, it follows that $M \in \mathfrak{z}(\mathfrak{sch}_2)$. Hence $\mathfrak{z}(\mathfrak{sch}_2) = \langle M \rangle$.

Now assume that $\mathfrak{z}(\mathfrak{sch}_{n-1}) = \langle M \rangle$. Note that the basis of $\mathfrak{sch}_n$ is

$$B_{\mathfrak{sch}_n} = B_{\mathfrak{sch}_{n-1}} \bigcup \{x_{in}, 1 \le i \le n-1\} \bigcup \{x_n \frac{\partial}{\partial x_{n+1}}, x_n \frac{\partial}{\partial x_{n+2}}\}.$$

Let $X \in \mathfrak{sch}_n$. Then $X = Y + \sum_{i=1}^{n-1} a_i x_{in} + bx_n \frac{\partial}{\partial x_{n+1}} + cx_n \frac{\partial}{\partial x_{n+2}}$ for some $a, b, c \in \mathbb{R}$ with $Y \in \mathfrak{sch}_{n-1} \subseteq \mathfrak{sch}_n$. If $X \in \mathfrak{z}(\mathfrak{sch}_n)$, then for all $u \in \mathfrak{sch}_{n-1} \subseteq \mathfrak{sch}_n$ we have

$$0 = [X, u] = [Y, u] + \sum_{i=1}^{n-1} a_i [x_{in}, u] + b[x_n \frac{\partial}{\partial x_{n+1}}, u] + c[x_n \frac{\partial}{\partial x_{n+2}}, u].$$

Since $[Y, u]$ and $\sum_{i=1}^{n-1} a_i [x_{in}, u] + b[x_n \frac{\partial}{\partial x_{n+1}}, u] + c[x_n \frac{\partial}{\partial x_{n+2}}, u]$ share no common basis elements, it follows by linear independence that $[Y, u] = 0$ and $\sum_{i=n}^{n-1} a_i [x_{in}, u] + b[x_n \frac{\partial}{\partial x_{n+1}}, u] + c[x_n \frac{\partial}{\partial x_{n+2}}, u] = 0$. Hence $Y \in \mathfrak{z}(\mathfrak{sch}_{n-1})$, and thus $Y = mM$ for some $m \in \mathbb{R}$ by inductive hypothesis. So, we are reduced to $X = \sum_{i=1}^{n-1} a_i x_{in} + bx_n \frac{\partial}{\partial x_{n+1}} + cx_n \frac{\partial}{\partial x_{n+2}} + mM$. Again because $X \in \mathfrak{z}(\mathfrak{sch}_n)$, it follows that

$$\begin{aligned} 0 = [X, x_{1n}] &= \sum_{i=1}^{n-1} a_i [x_{in}, x_{1n}] + b[x_n \frac{\partial}{\partial x_{n+1}}, x_{1n}] + c[x_n \frac{\partial}{\partial x_{n+2}}, x_{1n}] + m[M, x_{1n}] \\ &= -a_2(x_{12}) - a_3(x_{13}) + \cdots - a_{n-1}(x_{1,n-1}) - b(x_1 \frac{\partial}{\partial x_{n+1}}) - c(x_1 \frac{\partial}{\partial x_{n+2}}). \end{aligned}$$

This implies by linear independence that $a_2, a_3, \cdots, a_{n-1}, b, c = 0$ and thus $X = a_1 x_{1n} + mM$. Repeating the previous calculations with x_{2n} gives $0 = [X, x_{2n}] = a_1[x_{1n}, x_{2n}] + m[M, x_{2n}] = a_1 x_{12}$, which implies that $a_1 = 0$ by linear independence. Hence $X = mM$ and thus $\mathfrak{z}(\mathfrak{sch}_n) \subseteq \langle M \rangle$. Now since by definition, $[M, X] = 0$ for all $X \in \mathfrak{sch}_n$, it follows that $M \in \mathfrak{z}(\mathfrak{sch}_n)$. Hence $\mathfrak{z}(\mathfrak{sch}_n) = \langle M \rangle$.

□

3.2 The Center of the Galilei Lie Algebra

The basis of the centrally extended Galilei Lie Algebra is given by

$$\{M, x_{ij}, \beta_n, x_i \frac{\partial}{\partial x_{n+1}},\ x_i \frac{\partial}{\partial x_{n+2}};\ \ 1 \le i, j \le n\},$$

so it is a subalgebra of the centrally extended Schrödinger Lie algebra [2].

Proposition 3.3. The center of the centrally extended Galilei Lie algebra $\mathfrak{gal}_n$ with mass M coincides with the subalgebra generated by its mass; that is

$$\mathfrak{z}(\mathfrak{gal}_n) = \langle M \rangle \quad \text{for all} \;\; n \geq 1.$$

Proof. We briefly outline the proof as it is similar to the Schrödinger Lie algebra case. Again we proceed by induction on n starting from $n = 2$ for clarity, since $\mathfrak{gal}_1$ is the only case where there are no rotation generators. The calculation of $\mathfrak{z}(\mathfrak{gal}_1)$ is analogous and easier than $\mathfrak{z}(\mathfrak{gal}_2)$. The basis of $\mathfrak{gal}(2)$ is $B_{\mathfrak{gal}_2} = \{M, x_{12}, \beta_2, x_1\frac{\partial}{\partial x_3}, x_2\frac{\partial}{\partial x_3}, x_1\frac{\partial}{\partial x_4}, x_2\frac{\partial}{\partial x_4}\}$. Let $X \in \mathfrak{z}(\mathfrak{gal}_2)$. Then

$$X = ax_{12} + bx_1\frac{\partial}{\partial x_3} + cx_2\frac{\partial}{\partial x_3} + dx_1\frac{\partial}{\partial x_4} + ex_2\frac{\partial}{\partial x_4} + f\beta_2 + gM$$

for some $a, b, c, d, e, f, g \in \mathbb{R}$. Using the fact that $[X, x_{12}] = 0$ and the linearity of the bracket operation, we have $b(-x_2\frac{\partial}{\partial x_3}) + c(x_1\frac{\partial}{\partial x_3}) + d(-x_2\frac{\partial}{\partial x_4}) + e(x_1\frac{\partial}{\partial x_4}) = 0$ which implies that $b, c, d, e = 0$ by linear independence. So $X = ax_{12} + f\beta_2 + gM$. Repeat with $[X, x_1\frac{\partial}{\partial x_3}] = 0$ to have $ax_2\frac{\partial}{\partial x_3} - f(x_1\frac{\partial}{\partial x_4}) = 0$, and thus $a, f = 0$ by linear independence. Hence $X = gM$ and thus $\mathfrak{z}(\mathfrak{gal}_2) \subseteq \langle M \rangle$. Since by definition $[M, X] = 0$ for all $X \in \mathfrak{gal}_2$, it follows that $M \in \mathfrak{z}(\mathfrak{gal}_2)$. Hence $\mathfrak{z}(\mathfrak{gal}_2) = \langle M \rangle$.

Now assume that $\mathfrak{z}(\mathfrak{gal}_{n-1}) = \langle M \rangle$ and let $X \in \mathfrak{z}(\mathfrak{gal}_n)$. Since the basis of $\mathfrak{gal}_n$ is

$$B_{\mathfrak{gal}_n} = B_{\mathfrak{gal}_{n-1}} \cup \{x_{in}, 1 \leq i \leq n-1\} \cup \{x_n\frac{\partial}{\partial x_{n+1}}, x_n\frac{\partial}{\partial x_{n+2}}\}$$

we have

$$X = Y + \sum_{i=1}^{n-1} a_i x_{in} + bx_n\frac{\partial}{\partial x_{n+1}} + cx_n\frac{\partial}{\partial x_{n+2}}$$

for some $a, b, c \in \mathbb{R}$ with $Y \in \mathfrak{gal}_{n-1} \subseteq \mathfrak{gal}_n$. So for $u \in \mathfrak{gal}_{n-1} \subseteq \mathfrak{gal}_n$,

$$[X, u] = [Y, u] + \sum_{i=i}^{n-1} a_i[x_{in}, u] + b[x_n\frac{\partial}{\partial x_{n+1}}, u] + c[x_n\frac{\partial}{\partial x_{n+2}}, u] = 0$$

which gives $[Y, u] = 0$ since $[Y, u]$ and $\sum_{i=n}^{n-1} a_i[x_{in}, u] + b[x_n\frac{\partial}{\partial x_{n+1}}, u] + c[x_n\frac{\partial}{\partial x_{n+2}}, u]$ are linear combinations of different basis elements. Therefore $Y \in \mathfrak{z}(\mathfrak{gal}_{n-1})$ and thus $Y = mM$ for some $m \in \mathbb{R}$ by inductive hypothesis. So $X = mM + \sum_{i=n}^{n-1} a_i x_{in} + bx_n\frac{\partial}{\partial x_{n+1}} + cx_n\frac{\partial}{\partial x_{n+2}}$ Again because $X \in \mathfrak{z}(\mathfrak{gal}_n)$, we have $[X, x_{1n}] = 0$. This implies after applying the bracket operations and linearity that

$$-a_2(x_{12}) - a_3(x_{13}) + \cdots - a_{n-1}(x_{1,n-1}) - b(x_1\frac{\partial}{\partial x_{n+1}}) - c(x_1\frac{\partial}{\partial x_{n+2}}) = 0$$

and thus $a_2, \alpha_3, \cdots, a_{n-1}, b, c = 0$ by linear independence. So We are reduced to $X = mM + a_1x_{1n}$. Repeating the process on $[X, x_{2n}] = 0$ gives $a_1x_{12} = 0$ implying by linear independence that $a_1 = 0$. Hence $X = mM$ and thus $\mathfrak{z}(\mathfrak{gal}_n) \subseteq \langle M \rangle$. Now since by definition, $[M, X] = 0$ for all $X \in \mathfrak{gal}_n$, it follows that $M \in \mathfrak{z}(\mathfrak{gal}_n)$. Hence $\mathfrak{z}(\mathfrak{gal}_n) = \langle M \rangle$.

□

3.3 The Center of the Harmonic Oscillator Lie Algebra

The Harmonic Oscillator Lie algebra $\mathfrak{ho}_n$ is also a subalgebra of the Schrödinger Lie algebra. Its basis [3] is given by

$$\Big\{M, \alpha_n, x_i\frac{\partial}{\partial x_{n+1}},\ x_i\frac{\partial}{\partial x_{n+2}};\ \ 1 \leq i \leq n\Big\}.$$

Proposition 3.4. The center of the centrally extended harmonic Oscillator Lie algebra $\mathfrak{ho}_n$ with mass M coincides with the subalgebra generated by its mass; that is

$$\mathfrak{z}(\mathfrak{ho}_n) = \langle M \rangle \ \text{ for all } \ n \geq 1.$$

Proof. The proof is also done by induction on n and uses the same calculation techniques as in the previous cases. □

3.4 The Center of the Heinsenberg-Weyl Lie Algebra

The Heinsenberg-Weyl Lie algebra $\mathfrak{hw}_n$ is also a subalgebra of the Schrödinger Lie algebra. Its basis [5] is given by

$$\Big\{M, x_i\frac{\partial}{\partial x_{n+1}},\ x_i\frac{\partial}{\partial x_{n+2}};\ \ 1 \leq i \leq n\Big\}.$$

Proposition 3.5. The center of Heinsenberg-Weyl Lie algebra $\mathfrak{hw}_n$ with mass M coincides with the subalgebra generated by its mass; that is

$$\mathfrak{z}(\mathfrak{hw}_n) = \langle M \rangle \quad \text{for all } \ n \geq 1.$$

Proof. The proof is similar to the previous cases. □

References

[1] Barannyk, L., On the Classification of Subalgebras of the Galilei Algebras, *Nonlinear Math. Phys.* **2**, 3-4(1995), 263-268.

[2] Biyogmam, G., R., Leibniz Homology of the Galilei Algebra, *J. Math. Phy.*, **54**, 073514 (2013).

[3] Biyogmam, G., R., *Leibniz Homology of the Harmonic Oscillator Lie Algebra,* 2013, preprint.

[4] Sternberg, S., *Group Theory and Physics*, Cambrigde University Press, 1999.

[5] Feinsilver, Ph., Kocik, J., Schott, R., Representation of the Schrödinger Algebra and Appell Systems, *Fortschritte der Physik*, **52**, 4(2004),343-359.

[6] Humphreys, J. E., *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York-Heidelberg-Berlin, 1972.

[7] Kirillov, A., *An Introduction to Lie Groups and Lie Algebras,* Cambridge University Press, 2008.

[8] Mohlenkamp, M., J., A Center of Mass Principle for the Multiparticle Schrödinger Equation, *J. Math. Phy.*, **51**, 022112 (2010).

[9] Sternberg, S., *Group Theory and Physics*, Cambrigde University Press, 1999.