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ANTI-CLOAKING: THE MATHEMATICS OF DISGUISE

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Abstract: Recent developments in cloaking, the ability to selectively bend electromagnetic waves so as to render an object invisible, have been abundant. Based on cloaking principles, we will describe several ways to mathematically disguise objects in the context of electrical impedance imaging. Through the use of a change-of-variables scheme we show how one can make an object appear enlarged, translated, or rotated by surrounding it with a suitable "metamaterial," a man-made material that selectively redirects current. Analysis of eigenvectors and eigenvalues, which describe how current flows, follow. We prove that in order to disguise an object, a metamaterial must encompass both the subdomain and its disguised version, and discuss the consequences. Finally, we briefly explore how electricity is just a springboard to potential applications.

1. Introduction

Popularized by the Star Trek series as a Romulan cover-up, cloaking has been recently seen on screen in both The Lord of the Rings and Harry Potter movies. But this science fiction is now being transformed into science fact. The scientific and mathematical communities are busy describing and trying to build different types of cloaks; for a general overview, see Greenleaf, et. al [4].

Instead of the principles of cloaking using the full Maxwell's equations for electromagnetic energy, we'll stick to the simpler equations that govern "impedance imaging,", described below.

To begin, we first define a bit of terminology. There are two types of electrical conductors that we will be considering: isotropic and anisotropic. An *anisotropic* conductor has directionally dependent properties present in the material—the material conducts electricity more easily in some directions that other. An *isotropic* conductor has no directional properties. A material is *homogenous* if the physical properties are the same at each point in space, and *nonhomogeneous* otherwise.

We will use the following model for the simplest case, steady-state electrical conduction in an isotropic object Ω:

$$\mathbf{J} = \gamma \mathbf{E}, \tag{1.1}$$

where $\mathbf{J}$ denotes the current flux in Ω, $\mathbf{E}$ the electric field and γ is the *conductivity* conductivity of the object. If the material is homogeneous and isotropic, γ is a non-negative constant. However, if the material is nonhomogeneous, γ is a (scalar-valued) function of position. As γ grows, the electrical current flow $\mathbf{J}$ increases, for any fixed $\mathbf{E}$. Since $\mathbf{E} = -\nabla u$, where u is electrical potential in the object, equation (1.1) becomes

$$\mathbf{J} = -\gamma \nabla u.$$

Due to conservation of electrical charge we must have $\nabla \cdot \mathbf{J} = 0$ in Ω, that is, $\nabla \cdot \gamma \nabla u = 0$, which reduces to Laplace's equation, $\Delta u = 0$, when γ is constant.

For the anisotropic conduction case, there may be preferred directions for current flow at any given point in the object. For this reason, γ cannot be taken as a scalar. In this situation, there are maximum and minimum directions for the conductivity, and a common model is in which the corresponding directions are orthogonal to each other. When taking all of these conditions into account, equation (1.1) becomes

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ is a symmetric positive-definite matrix that may still depend on position. Note the isotropic case can be subsumed into this case, as $\sigma = \gamma \mathbf{I}$ where $\mathbf{I}$ is the identity matrix. See [3] for more on the derivation of the model for anisotropic conduction.

A *metamaterial* is an essential component of cloaking. This type of man-made material is macroscopically composed of at least two distinct materials that extend the range of electromagnetic patterns because the metamaterial cannot be found in nature. In fact, we construct anisotropic conductors with metamaterial that redirect the electrical energy around an object to be cloaked or disguised, which makes it appear in a different form to an outside observer. This is the essential property which we exploit throughout this paper to construct disguises and cloaks.

2. Impedance Imaging and the Forward Problem

To image the interior of an object, one can use a technique called Electrical Impedance Tomography (EIT). EIT involves applying an electrical current through electrodes attached to the boundary of an object. This induces an electric potential throughout the object, and current flows in accordance with equation (1.1) (or its anisotropic counterpart $\mathbf{J} = \sigma\mathbf{E}$.) The value of γ (or σ) alters the flow of current, and hence the value of the potential u on the exterior boundary of the object. One can then measure this boundary potential. By using this information—applied currents and measured boundary potentials—one can, at least in the isotropic conduction case, form an image of the interior electrical conductivity of the object. This technique has already found use in medical imaging; see [3] for images of a cross-section of the human torso obtained using this technology.

Let us quantify the ideas above a bit more carefully. Let Ω be a bounded, open connected set, or domain, in $\mathbb{R}^2$, though the techniques generalize to $\mathbb{R}^n$. We will use $\partial\Omega$ to represent the boundary of Ω. To solve the partial differential equations used in impedance imaging, we need some boundary conditions. These are usually either Dirichlet or Neumann. For Dirichlet boundary conditions, an input potential is given on $\partial\Omega$. For Neumann boundary conditions, we specify an input current on $\partial\Omega$. The latter is the type of boundary data we will assume—we specify the input current on $\partial\Omega$. Suppose the region Ω has an anisotropic (possibly nonhomogeneous) conductivity σ.

First we will consider the general problem where Ω is some anisotropic conductive material with conductivity σ. We apply an input current flux g to $\partial\Omega$. Then as remarked above, the resulting potential u in Ω satisfies

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega \tag{2.1}$$

with the Neumann boundary condition

$$(\sigma \nabla u) \cdot \mathbf{n} = g \text{ on } \partial\Omega. \tag{2.2}$$

Equations (2.1)-(2.2) determine the function u only up to an additive constant (see, e.g., [6]), but a unique solution can be obtained by, for example, adding the additional condition $\int_{\partial\Omega} u\, ds = 0$. When σ is known and g specified, equations (2.1)-(2.2) comprise the *forward problem*, a standard boundary value problem to be solved for the function u.

EIT is an example of an *inverse problem*: Here σ is considered unknown; we inject a known current flux g into $\partial\Omega$, measure the solution u to equations (2.1)-(2.2), and from this information (or more likely, many inputs currents and measured potentials) try to deduce σ. As mentioned above, in the case that σ is actually known to be isotropic ($\sigma = \gamma\mathbf{I}$) effective algorithms for estimating σ from this type of input-output data exist. In essence, one case use input current-output voltage data to image the interior conductivity of an object.

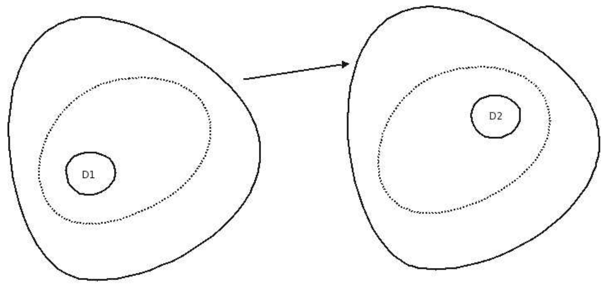

Figure 1. General mapping to anticloak, disguising D_1 as D_2

Our goal is to hide, or more generally "disguise" an object inside Ω, with respect to EIT. For the moment, suppose the object is a subdomain $D \subseteq \Omega$ such that the inward current flux satisfies:

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial D. \tag{2.3}$$

Thus D is an perfectly electrically insulating (non-conductive) subdomain that we can cloak or disguise.

If, for simplicity, we let $\gamma = 1$, then 2.1 becomes:

$$\Delta u = 0 \text{ in } \Omega \tag{2.4}$$

and 2.2 becomes:

$$\frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial\Omega. \tag{2.5}$$

We additionally specify a normalization $\int_{\partial\Omega} u\, ds = 0$ to nail down a unique solution to (2.4) and (2.5).

We will consider the case $\gamma = 1$ in Ω as the "base case,", that is, as the conductivity that an observer using EIT assumes exists inside Ω. Our goal is to surround D with a metamaterial—mathematically, an anisotropic conductor—that redirects current so as to make D appear differently. We want to use cloaking to disguise D, so as to make D look like something else, a new process which we call anticloaking (Figure 1)

3. Enlargements

3.1. **Introduction.** To begin our study of cloaking, let Ω be a bounded open region in $\mathbb{R}^2$ and D_1 a subdomain of Ω with $\overline{D_1} \subset \Omega$. Suppose that $\Omega \setminus D_1$ has "background" conductivity $\gamma \equiv 1$. Let D_2 be a subdomain of Ω with $\overline{D_2} \subset \Omega$. Let $x = (x_1, x_2)$ denote rectangular coordinates on

$\Omega \setminus D_1$ and $y = (y_1, y_2)$ rectangular coordinates on $\Omega \setminus D_2$. Let Φ be a C^2 mapping from $\overline{\Omega \setminus D_1}$ to $\overline{\Omega \setminus D_2}$ with C^2 inverse, and $D\Phi$ invertible on $\overline{\Omega \setminus D_1}$. Suppose also that Φ maps $\partial\Omega$ to $\partial\Omega$ and ∂D_1 to ∂D_2. Let u be the solution to equations (2.3), (2.4), and (2.5). Finally, define a function v on $\overline{\Omega \setminus D_2}$ as $v(y) = u(\Phi^{-1}(y))$. A standard computation, shown for example in [3], demonstrates the following lemma.

Lemma 1. *Under the conditions above the function v satisfies equations (2.1) and (2.2) with the condition $(\sigma \nabla u) \cdot \mathbf{n} = 0$ on ∂D_2, where*

$$\sigma(y) = \frac{(D\Phi(x))(D\Phi(x))^T}{det(D\Phi(x))}$$

with $x = \Phi^{-1}(y)$.

We now show how to use Lemma 1 to make an insulating circular region $D = B_\rho(0)$ (a ball of radius ρ centered at the origin) appear as a ball of radius 1/2 to an observer using EIT, in the case that Ω is the open unit disc in $\mathbb{R}^2$, though the principles are the same for every domain in $\mathbb{R}^2$ (Figure 2.) We use $x = (x_1, x_2)$ for rectangular coordinates and $\|x\| = \sqrt{x_1^2 + x_2^2}$ for the norm of x. To do this, we can apply a transformation Φ as above, a "push forward" map, to artificially enlarge D. Specifically, define

$$\Phi(x) = \frac{\Psi(r)x}{r}, \text{ where } \|x\| = r,$$

for $x \in \Omega \setminus B_{1/2}(0)$, where as above Φ and Φ^{-1} are invertible $\mathcal{C}^2$ mappings such that the Jacobian, $D\Phi$, is nonsingular on $\Omega \setminus B_{1/2}(0)$. We suppose that Φ maps $\Omega \setminus B_{1/2}(0)$ to $\Omega \setminus D$ with Φ the identity map in a neighborhood of $\partial\Omega$, and Φ maps the circle of radius 1/2 centered at the origin to ∂D.

Such a mapping can be obtained by taking

$$\Psi(r) = \begin{cases} \rho + \frac{r - \frac{1}{2}}{\frac{1}{2} + \delta} & \frac{1}{2} \le x \le \frac{1}{2} + \delta, \\ h(r) & \frac{1}{2} + \delta < x < \frac{1}{2} + 2\delta, \\ r & \frac{1}{2} + 2\delta \le x \le 1. \end{cases}$$

for $\delta \in (0, 1/4)$ and where $h(r)$ is a suitable smooth function to connect the two regions. Let $y = \Phi(x) \in \Omega \setminus D$ (with $y = (y_1, y_2)$) for $x \in \Omega \setminus B_{1/2}(0)$ and set $s = \|y\|$.

Let $u(x)$ be the potential in $\Omega \setminus D$ and v be defined by let $u(x) = v(\phi(x))$ (that is, $v(y) = u(\Phi^{-1}(y))$. In a neighborhood of the boundary $\partial\Omega$, $\frac{1}{2} + 2\delta < r < 1$, we have $r = s$, so the potential functions u and v are equal. Under equation (2.4) and the definition of v, defined on

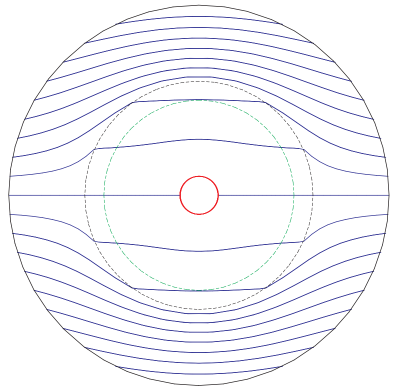

FIGURE 2. ϕ makes a circle of radius ρ appear like a circle of $\frac{1}{2}$ by redirecting the electrical current

the image of Φ, from Lemma 1 we conclude that v satisfies the partial differential equation (2.1) in $\Omega \setminus D$ with

$$\sigma(y) = \frac{D\Phi D\Phi^T}{|det(D\Phi)|}. \tag{3.1}$$

Here $\sigma(y)$ is a matrix that physically represents an anisotropic conductivity, for σ is clearly symmetric and positive-definite. Elsewhere we check that the Neumann boundary condition $\frac{\partial v}{\partial \mathbf{n}} = 0$ is satisfied, since

$$\begin{aligned} \frac{\partial v}{\partial \mathbf{n}} &= -\frac{\partial v}{\partial s}\bigg|_{s=\rho} \\ &= -\frac{\partial r}{\partial s}\frac{\partial u}{\partial r}\bigg|_{r=\frac{1}{2}} \\ &= (\frac{1}{2}+\delta)\frac{\partial u}{\partial r}\bigg|_{r=\frac{1}{2}} = 0. \end{aligned}$$

These calculations are independent of the input current, thus this anisotropic conductivity makes D look like a ball of radius $\frac{1}{2}$ for any input current g, making it an ideal disguise.

3.2. **Eigenvalues and Eigenvectors.** We can better understand the anisotropic conductivity matrix σ through its eigenvectors and eigenvalues. The eigenvectors of σ represent the directions of maximum and minimum conduction and the eigenvalues represent the magnitude of the conductivity in these directions. Since conductivity indicates a material's affinity to conduct electric current, by knowing these values, we can tell where and how much current will flow in all directions.

Given that σ is symmetric positive-definite, its eigenvectors are orthogonal and its eigenvalues are positive. These can be easily computed from $D\Phi$. A straightforward computation shows that

$$D\Phi = \left(\frac{\Psi'(r)}{r^2} - \frac{\Psi(r)}{r^3}\right)\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} + \left(\frac{\Psi(r)}{r}\right)I,$$

where I is the identity matrix. Since $D\Phi$ is clearly symmetric, from (3.1) we have $\sigma(y) = \frac{D\Phi^2}{|det(D\Phi)|}$, and the eigenvectors for $\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}$ are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ with corresponding eigenvalues $\|x\|$ and 0. These are also the eigenvectors for σ, but with shifted eigenvalues

$$\gamma_1 = \frac{r\Psi'(r)}{\Psi(r)}$$

$$\gamma_2 = \frac{\Psi(r)}{r\Psi'(r)}.$$

If we look at the innermost region, $\frac{1}{2} \le r \le \frac{1}{2} + \delta$, we find

$$\gamma_1 = \frac{r}{\rho(\frac{1}{2}+\delta) + r - \frac{1}{2}}$$

and

$$\gamma_2 = \frac{\rho(\frac{1}{2}+\delta) + r - \frac{1}{2}}{r} = \frac{1}{\gamma_1}.$$

These eigenvalues, γ_1 and γ_2, correspond to orthogonal eigenvectors in the outward normal (v_1) and tangential (v_2) directions respectively for $\frac{1}{2} \le r \le \frac{1}{2} + \delta$. When $r = \frac{1}{2}$,

$$\gamma_1 = \frac{1}{2\rho(\frac{1}{2}+\delta)}$$

which approaches ∞ when ρ approaches 0^+ and

$$\gamma_2 = 2\rho(\frac{1}{2}+\delta)$$

which approaches 0 when ρ approaches 0^+. Thus, the maximal conductivity is in the outward normal direction and the minimal conductivity is in the tangential direction. The eigenvalues for the region $\frac{1}{2} + 2\delta \le r \le 1$ are $\gamma_1 = \gamma_2 = 1$, since in this region Φ is the identity map.

We can also examine the eigenvalues for σ in terms of s, where $s = \Psi(r)$. By simple algebra, $r = (s-\rho)(\frac{1}{2}+\delta)+\frac{1}{2}$. For the region $\rho \le s \le \frac{1}{2}$

(corresponding to $\frac{1}{2} \leq r \leq \frac{1}{2} + \delta$),

$$\gamma_1 = \frac{r\Psi'(r)}{\Psi(r)} = \frac{(s-\rho)(\frac{1}{2}+\delta)+\frac{1}{2}}{(\frac{1}{2}+\delta)s}$$

$$\gamma_2 = \frac{1}{\gamma_1} = \frac{(\frac{1}{2}+\delta)s}{(s-\rho)(\frac{1}{2}+\delta)+\frac{1}{2}}$$

When $s = \rho$,

$$\gamma_1 = \frac{1}{2\rho(\frac{1}{2}+\delta)}$$

$$\gamma_2 = 2\rho(\frac{1}{2}+\delta),$$

which were the same values found previously for $r = \frac{1}{2}$. When $s = \frac{1}{2}$,

$$\gamma_1 = \frac{(\frac{1}{2}-\rho)(\frac{1}{2}+\delta)+\frac{1}{2}}{(\frac{1}{2}+\delta)}$$

$$\gamma_2 = \frac{(\frac{1}{2}+\delta)}{2(\frac{1}{2}-\rho)(\frac{1}{2}+\delta)+\frac{1}{2}},$$

which both approach 1 as δ and ρ approach 0^+. Notice that the eigenvalues are dependent on ρ since the map cannot be independent of ρ. All eigenvalues for a map of this nature will behave similarly in a neighborhood of the boundary as δ and ρ approach 0.

Some real-world examples of making a small hole appear large include magnification of an object, like a child using a magnifying glass to view an ant. Another application includes making an object appear more threatening, a possible survival mechanism. Currently, physicists around the world are quickly turning mathematical descriptions of cloaking into real-world objects at an astounding rate.

4. Disguising

Now that we have shown how to make a small hole appear large, we wanted to explore the possibility of disguising a subdomain of Ω as something else. Specifically, suppose we have an insulating subdomain D_1 contained in Ω, with background conductivity $\gamma \equiv 1$. A current g is input and we measure the resultant potential u on $\partial\Omega$. Since D_1 is insulating we have $\frac{\partial u}{\partial \mathbf{n}} = 0$ on ∂D_1. By "disguising" D_1 we mean that for any given $D_2 \subset \Omega$ we can find a suitable anisotropic conductivity σ defined on $\Omega \setminus D_2$ and a new potential v defined on $\Omega \setminus D_2$ such that v satisfies (2.1), (2.2) and

$$(\sigma \nabla v) \cdot \mathbf{n} = 0 \text{ on } \partial D_2.$$

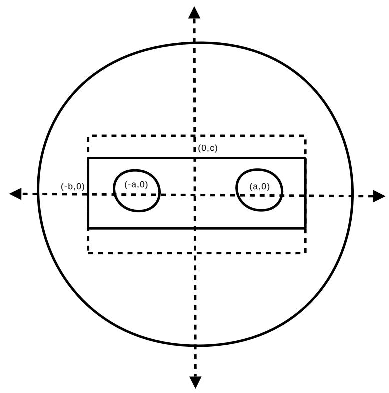

FIGURE 3. Mapping to move D_1 to D_2 by translation

Moreover, $\sigma = I$ and $v \equiv u$ in a neighborhood of $\partial\Omega$. In short, we surround D_2 with a suitable anisotropic conductor in such a way that for any input current g we obtain the same boundary voltage measurements as we do for the domain D_1 (surrounded by conductivity $\gamma = 1$.) Since the boundary condition $\frac{\partial u}{\partial n} = 0$ on ∂D_1 becomes $(\sigma \nabla_y v)\cdot\mathbf{n} = 0$ on ∂D_2 this means we need $\sigma = kI$ in a neighborhood of ∂D_2, where k is a scalar that may depend on position. So we need $\frac{(D\phi)^T(D\phi)}{|det(D\phi)|} = kI$, which means $(D\phi)^T(D\phi) = \tilde{k}I$. We conclude that $D\phi$ must be an orthogonal matrix in a neighborhood of ∂D_2.

We show how to achieve this in a specific case below.

4.1. **Translation.** To explore this endeavor, we began with the goal of translating and rotating a circle of radius R centered at $(-a, 0)$ in a continuous manner. To start, we wanted to translate our circle a fixed distance $d > 0$ inside Ω along the x-axis; refer to Figure 3. If $(b, 0)$ and $(0, c)$ are bounds to a rectangle that encloses D_1 and D_2 and lies within Ω, then the following maps accomplishes this. First, let

$$q(x, y) = G(x)(x + d, y) + (1 - G(x))(x, y)$$

where

$$G(x) = \begin{cases} 1 & -a - R \le x \le -a + R, \\ \frac{b-x}{b-(-a+R)} & -a + R < x < b, \\ \frac{b+x}{(-a-R+b)} & -b < x < -a - R. \end{cases}$$

Now define

$$\Phi(x, y) = t(y)q(x, y) + (1 - t)(x, y),$$

where

$$t(y) = \begin{cases} 1 & -c \leq y \leq c, \\ \frac{c+\delta-|y|}{\delta} & |c| < y < |c+\delta|, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < \delta$ and $(x, c+\delta) \in \Omega$ for all $x \in \Omega$. Notice that the metamaterial here is rectangular in nature and that both D_1 and D_2 are completely enclosed.

The general formula for the Jacobian of the translation map, $D\Phi$, is

$$\begin{pmatrix} \frac{\partial q_1}{\partial x} t + (1-t) & q_1 t_y + t\frac{\partial q_1}{\partial y} + x(1-t)_y \\ t\frac{\partial q_2}{\partial x} & q_2 t_y + t\frac{\partial q_2}{\partial y} + y(1-t)_y + (1-t) \end{pmatrix}$$

where $q(x,y) = (q_1(x,y), q_2(x,y))$. When $t = 1$, $-c \leq y \leq c$, so the Jacobian becomes

$$\begin{pmatrix} dG'(x)+1 & 0 \\ 0 & 1 \end{pmatrix},$$

when $t = 0$, we get the identity. When we look at the eigenvalues of the $\sigma = \frac{D\Phi(D\Phi)^T}{|det(D\Phi)|}$, we find when $t = 1, \lambda_1 = dG'(x)+1, \lambda_2 = \frac{1}{dG'(x)+1}$, when $t = 1, \lambda_1 = \lambda_2 = 1$, and when $t = \frac{\delta+c-|y|}{\delta}$, the formulas for these are much more complex. Instead, to illustrate how the eigenvalues change when the parameters of the map change, see the appendix. Notice as either d decreases or $|b+a-R|$ (if $-a+r < x < b$) or $|-a-R+b|$ (if $-b < x < -a-R$) increase, σ approaches 1. When $-b < x < b$ and $-c < y < c$, $t = 1$ and the eigenvalues are reciprocals.

Again, many applications are possible with this translation mapping and the rotation mapping that follows below. For example, since these allow an object to appear in a different location, programmers could use the technique in gaming.

4.2. **Rotation.** Next we wanted to disguise a circle inside of Ω by making the entire circle appear rotated by a fixed angle clockwise about the center of Ω. Note that in this case, the metamaterial is circular. A mapping that can accomplish this is:

$$\Phi(x,y) = (x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha),$$

where a general formula for α is

$$\alpha(r) = \begin{cases} \theta & 0 < r < \zeta \\ \frac{\theta}{\xi-\zeta}(1-\zeta) & \zeta \leq r \leq \xi \\ 0 & \xi < r \leq M, \end{cases}$$

where θ is the rotation angle clockwise, $0 < \zeta < \xi < M$, where M represents the radius of the metamaterial, and $r = \sqrt{x^2 + y^2}$. For future analysis, we pick a specific α :

$$\alpha(r) = \begin{cases} \theta & 0 < r < \frac{1}{2} \\ (3 - 4r)\theta & \frac{1}{2} \le r \le \frac{3}{4} \\ 0 & \frac{3}{4} < r \le 1, \end{cases}$$

By combining the functions for rotation and translation inside the unit disk, we are able to move a circle with radius less than 1 anywhere in Ω as long as $\left\| \partial\Omega - \partial D \right\|_\infty > 0$.

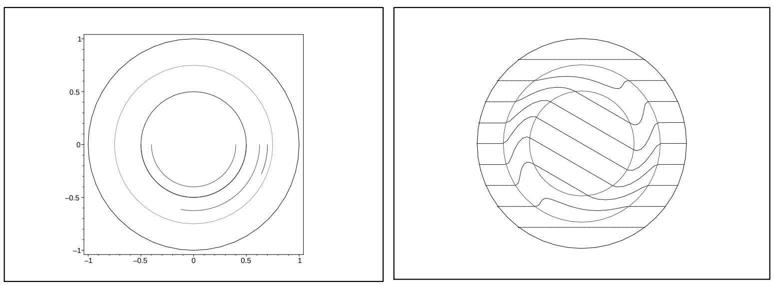

FIGURE 4. Rotation mapping depicting the rotational distortion between radius $\frac{1}{2}$ and $\frac{3}{4}$ (left).Rotation mapping of the redirected electrical current given a rotation angle of $\frac{\pi}{3}$ (right).

From the Jacobian of the rotation map we can compute the eigenvalues of σ, since $\sigma = \frac{(D\Phi)(D\Phi)^T}{\det D\Phi}$. For our $\Phi(x, y)$, we have that for $\alpha = \theta$,

$$D\Phi = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -cos(\theta) & sin(\theta) \end{pmatrix}$$

and the eigenvalues of σ are $\lambda_1 = \lambda_2 = 1$. For $\alpha = 0, D\Phi = I$ so $\lambda_1 = \lambda_2 = 1$. The eigenvectors for both are orthonormal, with the former depending on θ. Observations for eigenvalues and eigenvectors of σ appear in the appendix.

Notice that the second eigenvalue increases for $r = \frac{1}{2}$ as θ changes from 0 to 2π, indicating more distortion of the inner boundary of the metamaterial.

4.3. **General Observations about disguising maps.** Firstly, exploration of the eigenvalues of the translation and rotation mappings led us to the following general theorem:

Theorem 1. *Let $C = \frac{AA^t}{|detA|}$, where $A \in GL(2)$. Then the eigenvalues of C are reciprocals.*

Proof.
$$\frac{AA^t}{|detA|} = \frac{AA^t}{\sqrt{\det(AA^t)}} = \frac{B}{\sqrt{detB}},$$
where $B = AA^t$, since $\det(AA^T) = (\det A)^2$. Thus, B is a positive symmetric definite matrix. Let $\frac{B}{\sqrt{detB}}v_1 = \mu_1 v_1, \frac{B}{\sqrt{\det B}}v_2 = \mu_2 v_2$ and $Bv_1 = \lambda_1 v_1, Bv_2 = \lambda_2 v_2$. Thus,
$$\lambda_1 = \mu_1 \sqrt{\det B}$$
and
$$\lambda_2 = \mu_2 \sqrt{\det B}.$$
Since $\det B = \lambda_1 \lambda_2, \det B > 0$, and
$$\lambda_1 > 0, \lambda_2 > 0,$$
then $(\det B)^2 = \lambda_1^2 \lambda_2^2$, and
$$\mu_1^2 = \frac{\lambda_1^2}{\det B} = \frac{\det B}{\lambda_2^2} = \frac{1}{\mu_2^2}.$$
Therefore, since μ_1 and μ_2 are both positive,
$$\mu_1 = \frac{1}{\mu_2}.$$
□

This theorem is analogous to n dimensions as well.

Corollary 1. *Let μ_i be the i^{th} eigenvalue of $\frac{B}{\sqrt{\det B}}$, and λ_i be the i^{th} eigenvalue of B. Then $\mu_i = \frac{1}{\mu_j \lambda_1 \cdots \hat{\lambda_i} \cdots \hat{\lambda_j} \cdots \lambda_n}$, where $\hat{\lambda_i}$ means that it has been removed.*

Proof. This result follows immediately applying the same technique as Theorem 1. □

So we are now able to make a circle of radius R appear anywhere within a neighborhood of the boundary of the unit disc with translations and rotations. This analysis not only works on the disk, but on any path connected domain in $\mathbb{R}^2$. There are limitations in what we can make D look like. Since the map is assumed to be continuous, it maps a connected set to a connected one, so, for instance, we cannot make one circle look like two. One of the major discoveries of this research are the corollaries to the following theorem that describe some of the limitations to anticloaking. If we want to disguise D_1 as D_2, we must surround both with a suitable metamaterial. However, there are still plenty of applications in spite of this seeming handicap; see for example [7].

Though proven in the context of $\mathbb{R}^2$, the following lemma applies to any bounded open connected set (domain) in $\mathbb{R}^n$. The subdomains are open sets and u and v are always assumed to be $\mathcal{C}^2$. The main point of the lemma is that without the presence of a metamaterial (a suitable anisotropic conductor) an insulating subdomain D_1 will always be distinguishable from another insulating subdomain D_2 with just one nonzero input flux and measured boundary voltage.

Lemma 2. *Let Ω be a domain in $\mathbb{R}^2$ and let D_1 be a subdomain. A nonzero current flux g is input such that the potential u defined in $\Omega \setminus D_1$ satisfies*

$$\Delta u = 0 \text{ in } \Omega \setminus D_1,$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial\Omega, \text{ and}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial D_1.$$

Let D_2 be a subdomain of Ω. The same current flux g is input such that the potential v defined in $\Omega \setminus D_2$ satisfies:

$$\Delta v = 0 \text{ in } \Omega \setminus D_2,$$

$$\frac{\partial v}{\partial \mathbf{n}} = g \text{ on } \partial\Omega, \text{ and}$$

$$\frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial D_2.$$

If $u = v$ on some open portion of $\partial\Omega$ then $D_1 = D_2$.

Proof. Assume towards a contradiction that $u = v$ on some open portion of $\partial\Omega$ and that $D_1 \neq D_2$. We know $u = v$ everywhere in $\Omega \setminus (D_1 \cup D_2)$ by unique continuation, since u and v are both harmonic and have the same Cauchy (Dirichlet and Neumann) data on an open portion of $\partial\Omega$. Their normal derivatives also agree by unique continuation. Without loss of generality, let D be a connected component of $D_2 \setminus D_1$. Thus ∂D consists of portions of ∂D_1 and ∂D_2. Additionally,

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial D_1;$$

and

$$\frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial D_2,$$

therefore

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial D.$$

Since $\Delta u = 0$ in D, and $\frac{\partial u}{\partial \mathbf{n}} = 0$ on ∂D we must conclude that u is constant in D (see, e.g., [6]). This forces $u = c$ everywhere in $\Omega \setminus D_1$, (again, see [6]) which implies $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, a contradiction since g is nonzero. □

Thus according to Lemma 2 any nonzero flux g must yield different boundary potentials for $\Omega \setminus D_1$ and $\Omega \setminus D_2$; if the boundary potentials agree, on *any* portion of $\partial\Omega$, then it must be that $D_1 = D_2$.

The next two lemmas show that if a region D_1 is to be disguised as D_2 then the anisotropic conductor (the metamaterial) that surrounds D_1 must in fact fully enclose D_2.

Lemma 3. *Let D_1, D_2 be open subsets of Ω with $D_1 \cap D_2 = \emptyset$ and let M be a subset of Ω and*

$$\sigma(x,y) = \begin{cases} 1 & \text{if } (x,y) \in M \\ \sigma_0(x,y) & \text{if } (x,y) \notin M \end{cases}$$

where $\overline{D_1} \subset M \subseteq \Omega$, and

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega \setminus D_1$$
$$(\nabla u) \cdot \mathbf{n} = g \text{ on } \partial\Omega,$$
$$(\sigma \nabla u) \cdot \mathbf{n} = 0 \text{ on } \partial D_1$$
$$\Delta v = 0 \text{ in } \Omega \setminus D_2,$$
$$(\nabla v) \cdot \mathbf{n} = g \text{ on } \partial\Omega,$$
$$(\nabla v) \cdot \mathbf{n} = 0 \text{ on } \partial D_2$$

where $g \neq 0$. Let $u = v$ on some open portion of $\partial\Omega$. If $M \cap D_2 = \emptyset$, then $D_2 = \emptyset$.

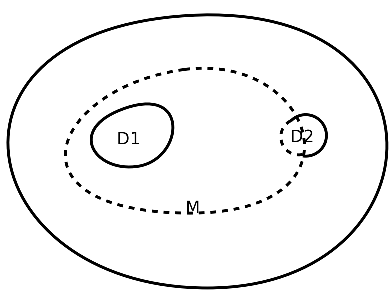

FIGURE 5. D_2 partially contained in metamaterial

Proof. We know $u = v$ in $\Omega \backslash M$ and so under the assumption $M \cap D_2 = \emptyset$ we have

$$(\nabla v) \cdot \mathbf{n} = 0 = (\nabla u) \cdot \mathbf{n} \text{ on } \partial D_2.$$

This implies that $(\nabla u) \cdot \mathbf{n} = 0$ on ∂D_2, and note that $\triangle u = 0$ in D_2. Thus as argued above u must be constant on D_2, and hence everywhere in $\Omega \setminus M$ a contradiction to the nonzero input flux. We conclude that $D_2 = \emptyset$.

□

From Lemma 3 $M \cap D_2$ has a non-empty interior. In fact, D_2 must be contained in M if D_1 is to be disguised as D_2 for all possible input fluxes, as the next theorem states, and as is shown in Figure 6.

Theorem 2. *Let u and v be as in Lemma 3, and suppose in particular that $u = v$ on $\partial\Omega$ for EVERY input flux g. Then $D_2 \subseteq M$.*

Proof. First, let $\mathbf{n_2}$ be the outward normal on D_2. Assume towards a contradiction that $D_2 \nsubseteq M$, as in Figure 5, and note then that $D = D_2 \setminus M$ has non-empty interior. However, by Lemma 3 we have $M \cap D_2 \neq \emptyset$, so the boundary ∂D consists of pieces of ∂M and pieces of ∂D_2. Then from unique continuation we conclude that $u \equiv v$ in $\Omega \setminus (M \cup D_2)$. In particular, since $\frac{\partial v}{\partial \mathbf{n_2}} = 0$ on ∂D_2 we have that

$$\frac{\partial u}{\partial \mathbf{n}_2} = 0$$

on the curve $S = \partial D_2 \cap (\Omega \setminus M)$. In short, we conclude that for every input flux g the function u that satisfies the relevant equations of Lemma 3 also satisfies $\frac{\partial u}{\partial \mathbf{n_2}} = 0$ on S. However, the techniques in [2] show that one can always find some non-zero flux g so that $\frac{\partial u}{\partial \mathbf{n_2}} \neq 0$ at some point in S. We conclude that $D_2 \subset M$. □

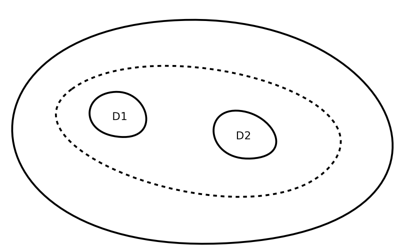

FIGURE 6. D_2 must be completely contained in the metamaterial

5. Thermal Imaging

5.1. **Introduction.** To begin our study of thermal imaging, we asked the questions, "Do the same principles of electrical imaging apply to thermal imaging?" and "Could we model the heat equation and use this to disguise?" To determine the answers, we replicate the previous two-dimensional example on the open unit disc in $\mathbb{R}^2$, which we denote by Ω, and examine a thermally insulating subdomain D. An input heat flux, g, is applied to Ω and we measure the resultant temperature u on $\partial\Omega$. The heat equation, unlike electrical imaging, is not steady-state and therefore has a time component. In the simplest case, under the usual modeling assumptions and after rescaling the temperature, $u(x, y, t)$, must satisfy the following equations:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \tag{5.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial\Omega \tag{5.2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial D, \tag{5.3}$$

along with an initial condition, in our case, $u(x, y, 0) = 0$, an initial temperature of 0.

We assume that the input flux is periodic, of the form $g(x, y, t) = g_0(x, y)e^{i\omega t}$. In this case (after transients have died out) we have $u(x, y, t) = u(x, y)e^{i\omega t}$ where equation (5.1) becomes

$$i\omega u - \Delta u = 0, \tag{5.4}$$

while (5.2) becomes

$$\frac{\partial u}{\partial \mathbf{n}} = g_0 \text{ on } \partial\Omega, \tag{5.5}$$

and (5.3) remains the same. We will call equation (5.4) the "periodic heat equation."

5.2. **Formulation and Meaning of the Heat Equation.** In what follows we will confine our attention to the case in which D is a disk of radius $R < 1$ centered at the origin, so that $\Omega \setminus D$ is and annulus. The periodic heat equation (5.4) can be written in (r, θ) polar coordinates as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} - i\omega u = 0 \tag{5.6}$$

We can solve this on the annulus using separation of variables, which we will use in later analysis. Let $u(r, \theta) = f(r)g(\theta)$, then divide the heat equation by $f(r)g(\theta)$, multiply by r^2, and separate variables to find

$$r^2\frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} - i\omega r^2 = -\frac{g''(\theta)}{g(\theta)}. \tag{5.7}$$

A standard separation of variables argument shows that the solutions are $g(\theta) = e^{ik\theta}$ and $f(r) = C_1 I_{|k|}(ar) + C_2 K_{|k|}(ar)$ for $k \in \mathbb{Z}$, where I_k and K_k are the modified Bessel Functions, $a = -\frac{\sqrt{(2)}}{2}(1+i)\sqrt{\omega}$, and C_1 and C_2 are constants. For a review of modified Bessel functions, important identities, and approximations used in this section, please see the appendix. Due to linearity, linear combinations are also solutions. The general solution to equation (5.4) is of the form

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} c_k I_{|k|}(ar)e^{ik\theta} + d_k K_{|k|}(ar)e^{ik\theta} \tag{5.8}$$

for constants c_k, d_k determined by the boundary conditions.

Note that on $\partial\Omega$ we have $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$, while on ∂D the derivative in the outward normal direction is $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}$. If we differentiate (5.8) with respect to r the boundary conditions (5.2) and (5.3) yield (after matching Fourier coefficients on the left and right)

$$c_k a I'_{|k|}(a) + d_k a K'_{|k|}(a) = g_k$$

$$c_k I'_{|k|}(aR) + d_k K'_{|k|}(aR) = 0$$

where $g_k = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-ik\theta}d\theta$. We find that

$$c_k = -\frac{g_k K'_{|k|}(aR)}{a(K'_{|k|}(a)I'_{|k|}(aR) - I'_{|k|}(a)K'_{|k|}(aR))}$$

$$d_k = \frac{g_k I'_{|k|}(aR)}{a(K'_{|k|}(a)I'_{|k|}(aR) - I'_{|k|}(a)K'_{|k|}(aR))}$$

a system of equations for the c_k, d_k.

More generally the heat equation may be written $c\rho\frac{\partial u}{\partial t} = \nabla\cdot(K_0\nabla u) + Q$, if we don't rescale/nondimensionalize variables, and include the possibility of a heat source Q. For the our case Q will equal 0. Thermal conductivity, or the ability of the material to conduct heat, appears in the K_0 term, where K_0 may be a matrix. The term of interest, however, is $c\rho$, a scalar, which corresponds to the specific heat and mass density. When expressed as shown, it is interpreted as volumetric heat capacity which describes the ability of a given volume of a substance to store internal energy while undergoing a given temperature change, but without undergoing a phase change. Note that if K_0 is a constant, the equation reduces to $\frac{\partial u}{\partial t} = k\nabla \cdot (\nabla u)$ where $k = \frac{K_0}{c\rho}$. In the special case that $k = 1$ we obtain equation (5.1).

If we want to cloak or anticloak using the heat equation, we will need to use a suitable change of variables. Let D_1 be a subdomain of Ω and let ϕ be an invertible map from $\overline{\Omega \setminus D_1}$ to $\overline{\Omega \setminus D_2}$, where D_2 is another subdomain. Assume ϕ satisfies the same differentiability conditions as in Section 3, and as there, that ϕ fixes a neighborhood of $\partial\Omega$. Let equations (5.3), 5.4) and (5.5) hold. Define a function v on the image of ϕ (in our case this v will be a temperature). Let $x = (x_1, x_2)$ be the rectangular coordinates of the domain and $y = (y_1, y_2)$ be the coordinates in the image of ϕ, as in the electrical case. Then $v(\phi(x)) = u(x)$.

Theorem 3. *Under the previous assumptions,*

$$\frac{i\omega v}{|D\phi|} - \nabla \cdot \sigma \nabla v = 0$$

on $\Omega \setminus D_2$, *where* $\sigma(y) = \frac{D\phi(x)(D\phi(x))^T}{|det(D\phi(x))|}$. *Note that here* σ *represents the thermal conductivity and* $\frac{1}{\|D\phi(x)\|}$ *is related to the change in mass density or specific heat due to the change in coordinates.*

Proof. This is essentially the proof of Lemma 1, but we give an outline here in the context of thermal cloaking, for the sake of completeness. A comparatively easy proof can be given using the divergence theorem. We know

$$\int_{\Omega\setminus D_1} r(x)(i\omega u - \Delta u)\, dx = 0 \tag{5.9}$$

where $r(x)$ is any $\mathcal{C}^1$ function on $\overline{\Omega \setminus D_1}$ with $r = 0$ on $\partial\Omega$ and $\bar{r}$ is defined on $\overline{\Omega \setminus D_2}$ by $\bar{r}(y) = r(\phi^{-1}(y))$.

Since $\Delta_x u = \nabla_x(r\nabla_x u) - \nabla_x r \cdot \nabla_x u$, after applying the divergence theorem, we get:

$$\int_{\Omega\setminus D_1} r(x) i\omega u dx - \int_{\partial(\Omega\setminus D_1)} r(x)\nabla_x u \cdot \mathbf{n} ds - \int_{\Omega\setminus D_1} \nabla_x r(x) \cdot \nabla_x u\, dx = 0 \tag{5.10}$$

By using the techniques from [3], and after changing variables as $y = \phi(x)$ and taking note of the boundary condition $\frac{\partial u}{\partial \mathbf{n}} = 0$ on ∂D_1, equation (5.10) becomes

$$\int_{\Omega\setminus D_2} \frac{\bar{r}(y) i\omega v(y)}{|det(D\phi)|} dy - \int_{\Omega\setminus D_2} \bar{r}(y)\nabla_y \cdot (\sigma(y)\nabla_y(v(y))dy$$
$$= \int_{\Omega\setminus D_2} \bar{r}(y)\left(\frac{i\omega v(y)}{|det(D\phi)|} - \nabla \cdot (\sigma(y)\nabla v)\right)dy = 0.$$

Since $\bar{r}(y)$ is arbitrary and this holds for any $\bar{r}$ that is $\mathcal{C}^1$, this forces $\frac{i\omega v}{|D\phi|} - \nabla \cdot (\sigma(y)\nabla v)$ to be identically zero in $\Omega \setminus D_2$. □

Now, we ask the question of how well we can cloak using the heat equation. To quantify this, we look at equation (5.4) on the annulus $B_1(0) \setminus B_\rho(0)$ and on the disk $B_1(0)$ (corresponding to using $\Omega = B_1(0)$ and $D = B_\rho(0)$ above). Specifically, we want to see how close the solutions get as ρ approaches zero.

Below we provide a proof of the following theorem, modulo a couple of unproven assertions.

Theorem 4. *Let u_ρ be the solution to the heat equation (5.4) with boundary conditions (5.5) and (5.3) on the annulus $B_1(0) \setminus B_\rho(0)$ and u_0 be the solution on the disk $B_1(0)$ with boundary condition (5.5). Then*

$$lim_{\rho \longrightarrow 0} \left\| u_\rho(1,\theta) - u_0(1,\theta) \right\|_{L^2(\partial B_1(0))} = 0$$

for any fixed $0 \le a < \infty$, where $a = \frac{\sqrt{2}}{2}(1+i)\sqrt{\omega}$.

Proof. The quantity we want to examine is (after a bit of algebra)

$$S(\rho) = \left\| u_\rho(1,\theta) - u_0(1,\theta) \right\|_{L^2(\partial B_1(0))} = \frac{1}{\alpha^4} \sum_{k=-\infty}^{\infty} |p_k(\rho)|^2 |g_k|^2$$

where the g_k are the Fourier coefficients of the boundary data and

$$p_k(\rho) = \frac{I'_{|k|}(\alpha\rho)}{I'_{|k|}(\alpha)} \frac{1}{\Delta} \tag{5.11}$$

where $\Delta = K'_{|k|}(\alpha\rho) I'_{|k|}(\alpha) - I'_{|k|}(\alpha\rho) K'_{|k|}(\alpha)$ with $\alpha = -\frac{1+i}{\sqrt{2}}\sqrt{\omega}$, $\omega > 0$. Our goal is to show that for any fixed $\omega > 0$ we have

$$\lim_{\rho \to 0^+} S(\rho) = 0.$$

Suppose we can show the following two assertions:

(1) For any fixed k we have

$$\lim_{\rho \to 0^+} p_k(\rho) = 0. \tag{5.12}$$

(2) For some $\rho_0 > 0$ and some M we have

$$|p_k(\rho)| \le M \tag{5.13}$$

for all $k \in \mathbb{Z}$ and all $0 < \rho < \rho_0$.

These would allow us to show that $S(\rho) \to 0$ as $\rho \to 0^+$, as follows. First, we know that $\sum_k |g_k|^2 < \infty$. This means that given any $\epsilon > 0$ we can find some N so that

$$\sum_{|k|>N} |g_k|^2 < \epsilon. \tag{5.14}$$

Given point (2) above we may conclude that

$$\sum_{|k|>N} |p_k(\rho)|^2 |g_k|^2 < \epsilon M^2$$

for $\rho < \rho_0$. If point (1) above holds then we may infer the existence of some ρ_1 such that

$$\sum_{|k|>N} |p_k(\rho)|^2 |g_k|^2 < \epsilon$$

for $\rho < \rho_1$ (making use of the fact that each summand limits to zero and the sum is finite.) Then for $\rho < \min(\rho_0, \rho_1)$ we find that

$$|S(\rho)| \le \frac{(M^2+1)\epsilon}{\alpha^4} = \epsilon'.$$

That is, $S(\rho)$ limits to zero.

However, we have not (yet) been able to prove assertions (5.12) and (5.13), though we are confident they are true. □

6. Conclusion and Future Direction

Through recent studies, the science and mathematics of cloaking possibilities have emerged. Our project gives insights to these discoveries in 2 dimensions. In our examples, we were able to using electrical impedance imaging to disguise by enlargement, relocation, and translation. An interesting continuation of our work includes the expansion of these examples into higher dimensions for practical use.

We were also able to cloak and provide physical interpretations for the thermal imaging case. For future direction, one could continue the studying of the thermal imaging case and attainably expand to other dimensions. Looking at this problem from the perspective of other types of imaging would also be of interest.

7. Appendix

7.1. **Bessel Functions.**

In this paper we discuss the modified Bessel functions $I_{\pm k}(a)$ and $K_k(a)$. Those are functions which are solutions to the following differential equation:

$$a^2 h_{aa} + a h_a - (a^2 + k^2) h = 0.$$

By relation 9.6.26 of [1],

$$e^{k\pi i}K_k'(a) = e^{k\pi i}e^{-\pi i}K_{k+1}(a) + \frac{k}{2}e^{k\pi i}K_k(a)$$

So

$$K_k'(a) = -K_{k+1}(a) + \frac{k}{2}K_k(a)$$

and

$$I_k'(a) = I_{k+1}(a) + \frac{k}{2}I_k(a)$$

Therefore:

$$\begin{aligned} h &= -K_{|k|}'(a)I_k(a) + K_k'(a)I_k'(a) \\ &= -(-K_{k+1}(a) + \frac{k}{2}K_k(a))I_k(a) + K_k(a)(I_{k+1}(a) + \frac{k}{2}I_k(a)) \\ &= K_{k+1}(a)I_k(a) + K_k(a)I_{k+1}(a) \\ &= \frac{1}{a} \end{aligned}$$

by relation 9.6.15. Note that $h \neq 0$ since $I_k(a)$ and $K_k(a)$ are linearly independent for all k [1].

Finally, we present the derivation for Laplace's equation on the disk, which is equivalent to the heat equation when a or ω equal 0. If $a = 0$, we get equations 1.2, 1.3, and 1.4. We must also assume that $\int_{\partial\Omega} g ds = 0$. Then,

$$u(1,\theta) = \sum_{k\in\mathbb{Z}} f_k r^{|k|} e^{ik\theta}, u_\rho(1,\theta) = \sum_{k\in\mathbb{Z}} \frac{f_k}{1+\rho^{2|k|}} e^{ik\theta}(r^{|k|} + \frac{\rho^{2|k|}}{r^{|k|}}),$$

where $f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}d\theta$

$$\begin{aligned} \int_0^{2\pi} (u_\rho(1,\theta) - u(1,\theta))^2 &= \sum_{k\in\mathbb{Z}} \left|(\frac{f_k}{1+\rho^{2|k|}}(r^{|k|} + \frac{\rho^{2|k|}}{r^{|k|}}) - f_k r^{|k|})\right|^2 \\ &= \sum_{k\in\mathbb{Z}} \left|\frac{f_k(\frac{1}{r^{|k|}} - r^{|k|})}{\frac{1}{\rho^{2|k|}} + 1}\right|^2 \end{aligned}$$

For $\rho < 1$ and $r \leq 1$, the above is less than or equal to

$$\sum_{k\in\mathbb{Z}} \left|\frac{f_k r^{|k|}}{\frac{1}{\rho^{2|k|}}}\right|^2,$$

which is a power series. By using a standard test (the root test), we find that it (and therefore the original series) converges uniformly for all $r < 1$.

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Square Metamaterial Eigenvalues and Eigenvectors For Given Cases

								Eigenvalues	Eigenvectors
x=0	y=0	a=1	b=2	c=1	R=1/4	delta=1/10	d=2	{11/3} {3/11}	{0,1} {1,0}
x=-3	y=0	a=1	b=2	c=1	R=1/4	delta=1/10	d=2	{1} {1}	{0,1} {1,0}
x=0	y=1.05	a=1	b=2	c=1	R=1/4	delta=1/10	d=2	{334.6723} {0.0030}	{0.9977, -0.0683} {0.0683,0.9977}
x=-1.5	y=0	a=1	b=2	c=1	R=1/4	delta=1/10	d=2	{11/3} {3/11}	{1,0} {0,1}
x=0	y=0	a=1	b=2	c=1	R=1/2	delta=1/10	d=0.1	{0.9600} {1.0417}	{1,0} {0,1}
x=0	y=0	a=1	b=2	c=1	R=1/2	delta=1/10	d=0.0001	{0.99997} {1.00003}	{1,0} {0,1}

Note: x, y, a, b, and c are shown in figure 4. Then, d is the distance from the original circle to the new circle. Delta is the distance from c (inside square) to the outer boundary of the metamaterial (outside square) and R is the radius of the circle to be translated.

Rotational Mapping Eigenvalues and Eigenvectors

	Values RR=1/2	Values RR=3/4	Vectors RR=1/2	Vectors RR=3/4
$\theta=0$	{1} {1}	{1} {1}	{0,1} {1,0}	{0,1} {1,0}
$\theta=\pi/4$	{0.2363} {4.2311}	{0.1348} {7.4168}	{-2.0570,1} {0.4862,1}	{2.1605,1} {-0.4629,1}
$\theta=\pi/3$	{0.1606} {6.2259}	{0.0849} {11.7847}	{-2.4952,1} {0.4008,1}	{1.0443,1} {-0.9576,1}
$\theta=\pi/2$	{0.0849} {11.7847}	{0.0414} {24.1652}	{-3.4329,1} {0.2913,1}	{0.2034,1} {-4.9158,1}
$\theta=\pi$	{0.0241} {41.4543}	{0.0110} {90.8154}	{-6.4385,1} {0.1553,1}	{-9.5297,1} {0.1049,1}
$\theta=3\pi/2$	{0.0110} {90.8154}	{0.0050} {201.8545}	{-9.5297,1} {0.1049,1}	{0.0704,1} {-14.2076}
$\theta=2\pi$	{0.0063} {159.9074}	{0.0028} {357.3030}	{-12.6455,1} {0.0791,1}	{-18.9025,1} {0.0529,1}

Note: in polar coordinates. Also, top eigenvalues increases and bottom eigenvalues decreases. This is due to distortion as the angle of rotation increases.