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Cops and Robbers on Oriented Graphs

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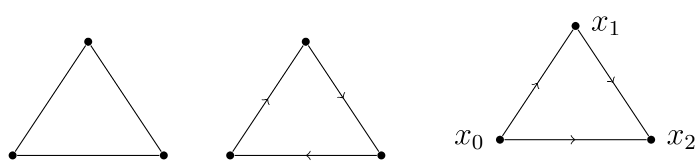
Abstract. Cops and Robbers is a turn-based game traditionally played on graphs. Similar to the classic children's game, one cop and one robber move along edges in a graph with the cop trying to catch the robber, and the robber trying to evade capture. In this article, we extend this game to oriented graphs. Although a complete characterization of 1-cop-win graphs is known, there is not yet a corresponding characterization for oriented graphs. Necessary conditions are described for an oriented graph to be 1-cop-win, and several results are provided toward finding sufficient conditions.

1 Introduction

Cops and Robbers is a combinatorial game that, in its current form, has been around since the 1980's [3]. Alspach [1] and Nowakowski and Winkler [5] first popularized the game, with the latter giving a complete characterization of 1-cop-win graphs. For the purposes of this article, we will restrict our attention to games played with only one cop. Specifically, given a finite graph G, the cop places himself on any vertex of the graph, followed by the robber, who then places himself on a different vertex. The game is turn-based, with the cop making the first move. The cop may either not move or traverse an edge that is incident to his starting position. Likewise, the robber can either not move or travel along any edge incident to his starting position. If the cop moves to the vertex that the robber currently resides on, the cop "captures" the robber and the game is over. More generally, a graph is *cop-win* if the cop can choose an initial position such that, for any initial position of the robber, the cop will eventually capture the robber. Similarly, a graph is *robber-win* if, given any initial starting point for the cop, there exists an initial position for the robber that always results in the robber evading capture.

Nowakowski [5] found a complete characterization of all graphs that are 1-cop-win, and Hill's thesis [4] discussed cop-win tournaments. However, a complete characterization of 1-cop-win directed graphs is still unknown. For the purposes of this article, we will restrict our attention to *oriented* graphs; that is, graphs with the property that every edge has exactly 1 orientation. Intuitively, the oriented graph game can be thought of as a game played on one-way streets (which the authors contend to be more realistic in an urban setting), whereas the traditional game is played on two-way streets.

To highlight how quickly oriented graphs differ from traditional graphs, consider the three vertex cycle C_3 shown in part (a) of Figure 1. In the standard graph case, C_3 is cop-win because the cop can start at any vertex and immediately capture the robber in the first move. However, gameplay changes when the edges are given an orientation. The orientation in part (b) of Figure 1 is robber-win since the robber can place himself at the vertex adjacent to the cop's vertex while part (c) is cop-win because when the cop places himself on the leftmost vertex he is able to capture the robber in the first move. Often, changing the direction of a single edge can drastically alter winning strategies and gameplay.



(a) *Cop-win* (b) *Robber-win* (c) *Cop-win*

Figure 1: Examples of oriented graphs with different outcomes.

In this article, we provide several results that make progress toward finding a complete characterization of all 1-cop-win oriented graphs. After some preliminary definitions in

Section 2, Section 3 highlights two main results. The first shows that if an oriented graph G is cop-win, then G contains exactly one vertex of in-degree 0 and the second proves that if G is a robber-win oriented graph, then G must contain a directed cycle. Finally, we conclude with necessary conditions for an oriented graph to be cop-win and provide several sufficient conditions for certain classes of oriented graphs.

2 Background

Most of the graph theory terminology used in this article is standard, and can be found in Chartrand and Lesniak [2]. A graph is *connected* if there is a path between any two vertices. A *directed edge* is defined by an ordered pair denoting direction, for instance, $\overrightarrow{x_0x_1}$ in Figure 1(c). In this example, we say x_0 is *adjacent to* x_1. The *in-degree* of a vertex v is the number of vertices adjacent to v, while the *out-degree* of a vertex is the number of vertices v is adjacent to. The *distance* between two vertices is the number of edges of the shortest path connecting the two vertices. A vertex x_1 is *reachable* from x_0 if there exists a directed path from x_0 to x_1. A *directed cycle* is a cycle with all edges oriented in the same direction. A *chordless directed cycle* is a directed cycle such that no two vertices of the directed cycle are connected by an edge that does not itself belong to the directed cycle.

3 Results

An oriented graph can quickly be determined to be robber-win if it lacks a vertex of in-degree 0 or has more than one such vertex.

Theorem 3.1. *If an oriented graph G is cop-win, then G contains exactly one vertex of in-degree 0, and every other vertex is reachable from this vertex.*

Proof. If the graph does not contain exactly one vertex of in-degree 0, then it either contains two or more, or none. If there are two or more vertices of in-degree 0, regardless of where the cop places himself, the robber will place himself on one of the other vertices of in-degree 0 and simply remain there indefinitely. The cop has no means of reaching the robber, so G is robber-win. If there are no vertices of in-degree 0, the robber places himself at a vertex that is adjacent to the cop's vertex. During each turn, the robber simply moves to the vertex the cop just left. Because edges in an oriented graph have only one direction, the cop can never reach the robber, so G is robber-win. Therefore, G must contain exactly one vertex of in-degree 0.

Now let x_0 be the unique vertex of in-degree 0. Then the cop must start on this vertex because if not, the robber will start there and remain there indefinitely. To show that every other vertex in the oriented graph is reachable from x_0, suppose not. Then there would have to exist a vertex, call it x_j, such that there was no path from x_0 to x_j. But then the robber can win by simply starting at x_j and staying at x_j for the entire game. Thus every vertex in the oriented graph must be reachable from x_0.

□

It should be noted that the converse of Theorem 3.1 does not hold. For example, Figure 2 has a single vertex of in-degree 0, x_0, but is in fact robber-win. To see why, note that the cop will start at x_0 and the robber can start on the far side of the directed cycle. Clearly, once the cop enters the directed cycle, the robber will always be able to stay at least one edge ahead of the cop. The converse of Theorem 3.1 does hold for several types of graphs however, including paths and trees.

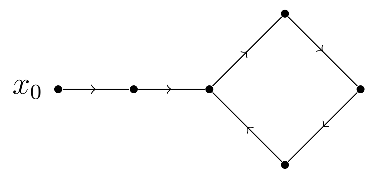


Figure 2: A robber-win oriented graph with exactly one vertex of in-degree 0.

Theorem 3.2. *The oriented path P_n on n vertices is cop-win if and only if it contains exactly one vertex of in-degree 0.*

Proof. The forward direction is Theorem 3.1. For the reverse direction, let us assume P_n contains exactly one vertex of in-degree 0. There are two cases to consider. In the first case (see for example, Figure 3(a)), the vertex of in-degree 0 is an end vertex. Then P_n must be a directed path, and the robber will be captured in at most $n-1$ moves. In the second case (Figure 3(b)), the vertex of in-degree 0 is not an end vertex. Then, P_n must be the union of two directed paths (otherwise P_n will have more than one vertex of in-degree 0). Since the cop will always be placed at the vertex of in-degree 0, the robber must be placed on one side or the other of the cop. The cop can then always pursue the robber in the appropriate direction, catching the robber in at most $n-2$ moves. □

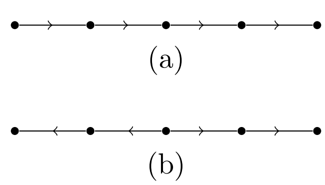


Figure 3: Two types of oriented paths.

Theorem 3.2 can be extended to finite oriented trees.

Theorem 3.3. *A finite oriented tree is cop-win if and only if it contains exactly one vertex of in-degree 0.*

Proof. Again, the forward direction is Theorem 3.1. For the reverse direction, the cop places himself at the vertex of in-degree 0 and the robber chooses a remaining vertex (see, for example, Figure 4). There is a unique directed path from the vertex of in-degree 0 to any other vertex in the tree. The cop follows the appropriate path toward the robber until the robber reaches an end vertex where he is then eventually caught. □

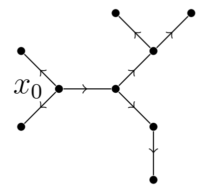


Figure 4: The vertex x_0 is the unique vertex of in-degree 0 which will always be the cop's initial position.

As illustrated in Figure 2, directed cycles play a fundamental role in determining whether a graph is cop-win or robber-win. For the remainder of this article, we will denote the vertices of any directed cycle C_n as $c_1, c_2, \ldots, c_n$, where c_i is adjacent to c_{j+1} for $1 \leq i \leq n-1$ and c_n is adjacent to c_1.

Theorem 3.4. *Let G be an oriented graph with only one vertex x_0 of in-degree 0 and the property that every other vertex is reachable from x_0. If G is robber-win, then G must contain a directed cycle.*

Proof. The cop will always start at the one vertex of in-degree 0. If he does not, then the robber can simply start there and never be forced to move. Since G is a finite oriented graph, hence containing only finitely many edges, in order for the graph to be robber-win, the robber must be able to evade capture forever. Thus the robber must eventually reach a vertex c_1 where he has previously been. If the robber's moves consisted of traversing edges $\overrightarrow{c_1c_2}, \overrightarrow{c_2c_3}, \ldots, \overrightarrow{c_nc_1}$, then the robber has traveled a directed cycle $c_1c_2 \cdots c_nc_1$. □

Notice that Theorem 3.2 and 3.3 are special cases of Theorem 3.4, since paths and trees do not contain any cycles. Unfortunately, the converse of Theorem 3.4 is also not true. For example, Figure 5 has a directed cycle but is cop-win. The cop simply starts at x_0 and will immediately capture the robber in his first move.

The above example illustrates that an oriented graph may contain directed cycles and yet remain cop-win, but every directed cycle must be appropriately dominated. We say a directed cycle is *cop-dominated* if the robber cannot avoid capture by entering the directed cycle by some path and then simply traveling repeatedly on the directed cycle. This notion of a cop-dominated directed cycle provides the cornerstone of cop-win oriented graphs.

Since identifying these directed cycles is an essential step in determining the winner of all oriented graphs, an algorithm for quickly identifying directed cycles in the graph is useful. One way to find the subgraph G' of an oriented graph G consisting of only directed cycles

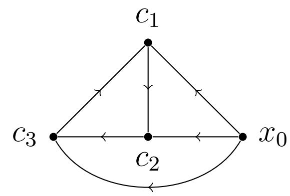
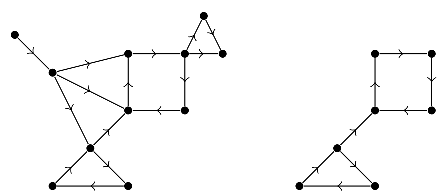


Figure 5: A cop-win oriented graph that contains a directed cycle.

and any edges between them is to iteratively remove all vertices of in-degree 0 and out-degree 0 and their corresponding incident edges (see, for example, Figure 6).



(a) G (b) G'

Figure 6: An oriented graph G and its subgraph G' consisting of only directed cycles and the edges between them.

In our attempt to provide a full characterization of all 1 cop-win oriented graphs, we have the following conjecture.

Conjecture 3.5. *An oriented graph G is cop-win if and only if G contains exactly one vertex of in-degree 0, every vertex is reachable from that vertex, and every directed cycle of G is cop-dominated.*

The forward direction follows from Theorem 3.1 and the definition of a cop-dominated cycle. The backward direction remains open, and remains open for even the simplest oriented graphs that contain just a single directed cycle (see Figures 5, 8, and 9 as examples). If we restrict our attention solely to a chordless directed cycle with no out-edges from any vertex on the directed cycle, then there are several conditions that must be satisfied in order for that directed cycle C_n to be cop-dominated. For notation, let C_n be a chordless directed cycle with vertices $c_1, c_2, \ldots, c_n$. Denote any non-cycle vertices that are adjacent to any vertex of C_n as $x_1, x_2 \ldots, x_m$. As always, x_0 will be the unique vertex of in-degree 0 in the graph, which may or may not be one of $x_1, x_2, \ldots, x_m$.

Theorem 3.6. *Let C_n be a chordless directed cycle in an oriented graph G with no out-edges from vertices on C_n to vertices not on C_n. The following conditions are necessary for C_n to be cop-dominated.*

(1) Let $x_1, x_2, \ldots, x_m$ denote the vertices of G that are not on C_n but are adjacent to at least one vertex of C_n. Then there is at least one edge $\overrightarrow{x_i c_j}$ for each j, $1 \leq j \leq n$.

(2) There exists a vertex x_i and index j such that x_i is adjacent to both c_j and c_{j+1}.

Proof. The first condition highlights the fact that there must exist vertices that "cover" every vertex of the chordless directed cycle. For if not, then there is at least one vertex of the cycle, say c_j, whose only in-edge is from c_{j-1}, the previous edge of the directed cycle. The robber can then simply start at c_j. Once the cop reaches c_{j-1}, the robber can proceed along the directed cycle C_n, always traveling one edge ahead of the cop, resulting in a robber-win graph.

The second condition states that not only must every vertex of the chordless directed cycle be covered, but also that the cop will be able to "jump ahead" and capture the robber traveling along the cycle at some point. For example, in Figure 7, we have the vertex x_1 that is adjacent to both c_1 and c_2 which allows the cop to potentially "jump ahead" to catch the robber. If such a vertex did not exist anywhere in the graph, then the robber can start the game at any c_j. Once the cop reaches a vertex adjacent to c_j, say x_j, the robber can proceed to c_{j+1} and, since x_j is not adjacent to c_{j+1}, the cop is then forced to follow along the directed cycle, resulting in a robber-win graph.

□

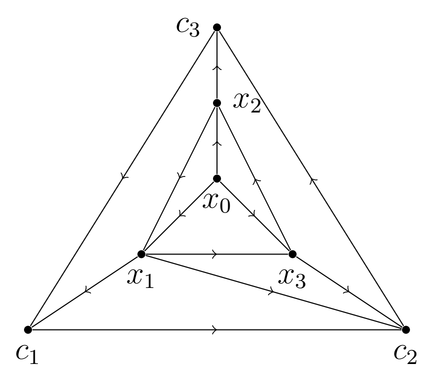
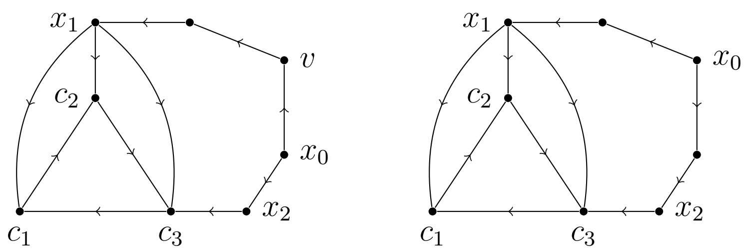


Figure 7: A cop-win oriented graph containing 3 vertices adjacent to the chordless directed cycle with 4 edges to the directed cycle.

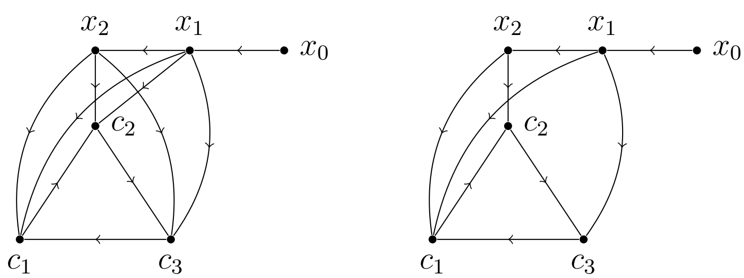
Together, these conditions provide restrictions on the structure of the graph regarding directed cycles. The edges incident to the x_i's are essential to the domination of a directed cycle, but the precise details of this relationship are not yet apparent. For example, Figure 8 illustrates the subtlety inherent in these structures, as switching the direction of the single edge $\overrightarrow{x_0 v}$ changes the graph from a cop-win graph to a robber-win graph. In Figure 8(b), the directed cycle is no longer cop-dominated because the robber can now initially place himself on x_2. The cop has to chase the robber to x_2, which eliminates the cop's path to x_1. The robber can then continually travel the directed cycle.



(a) *Cop-win* (b) *Robber-win*

Figure 8: Reversal of a single directed edge

From these necessary conditions we know that the minimum number of edges possible from the x_i's onto the directed cycle must be at least n for the graph to be cop-win. This lower bound can achieved, see Figure 5, which contains an oriented cycle of length three and three edges from x_0. In this situation, we see that the vertex x_0 is itself a "dominating vertex", or a vertex that ensures the directed cycle is cop-dominated. For a nontrivial example that uses more than just a single dominating vertex, Figure 7 contains an oriented cycle of length three and four edges from the x_i's onto the cycle.



(a) *Cop-win* (b) *Robber-win*

Figure 9: A cop-win oriented graph.

Conversely, we illustrate the maximum number of edges possible from the x_i's to the directed cycle where the graph remains robber-win. Let $c_1, ..., c_n$ be the directed cycle and let the structure of the x_i's consist of a path of length at least two from x_0, say $x_1, ..., x_m$. Now simply let $x_1, \ldots, x_m$ each be a dominating vertex of the directed cycle, that is, $\overrightarrow{x_i c_j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. This graph contains mn edges from the x_i's onto the cycle, which is in some sense the maximal number of edges. This oriented graph is clearly cop-win. However, removing edges $\overrightarrow{x_m c_j}$ and $\overrightarrow{x_{m-1} c_{j-1}}$ results in a robber-win oriented graph (in Figure 9 this corresponds to removing the edges $\overrightarrow{x_2 c_3}$ and $\overrightarrow{x_1 c_2}$). The robber's strategy would be to start at x_m, wait until the cop arrives at x_{m-1}, travel $\overrightarrow{x_m c_{j-1}}$, which forces the cop to traverse $\overrightarrow{x_{m-1} x_m}$, and subsequently the robber is now free to travel along the directed cycle with $\overrightarrow{c_{j-1} c_j}$. In the context of Figure 9, this corresponds to the robber starting at x_2, waiting for the cop to arrive at x_1, traveling $\overrightarrow{x_2c_2}$, and subsequently forcing the cop to travel $\overrightarrow{x_1x_2}$. The robber would then be free to travel $\overrightarrow{c_2c_3}$ and remain along the directed cycle. Thus we see that by removing a mere two edges from an "almost complete" graph, we can alter a cop-win oriented graph to robber-win.

The above examples show the complexity required of any characterization of 1-cop-win oriented graphs. There is clearly a relationship between the graph structure of the x_i's and the subsequent edges required from the x_i's onto any directed cycle, but the exact relationship is still unknown.

References

[1] Alspach, B. *Sweeping and Searching in Graphs: a Brief Survey*, Matematiche **59** (2006) 5-37.

[2] Chartrand, G. and Lesniak, L. *Graphs & Digraphs*, Chapman & Hall/CRC, (2005).

[3] Bonato, A. and Nowakowski, R. *The Game of Cops and Robbers on Graphs*, American Mathematical Society (2011).

[4] Hill, A. *Cops and Robbers: Theme and Variations*, Ph.D. Thesis, Dalhousie University, (2008).

[5] Nowakowski, R.J. and Winkler, P. *Vertex-to-Vertex Pursuit in a Graph*, Discrete Mathematics, **43** (2009) 237-251.