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Mathematical Modeling of Quadcopter Dynamics

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Introduction:

Recently, Google, Amazon and others are attempting to develop delivery drones for commercial use, in particular Amazon Prime Air promising 30 minute delivery. One type of commonly used drone proposed for such purposes is a quadcopter. Quadcopters have been around for some time with original development in the 1920's. They are popular now because they are mechanically simple and provide a good vehicle for unmanned flight. By controlling the speed of the four propellers, a quadcopter can roll, change pitch, change yaw, and accelerate.

This research will focus on the study of classical mechanics theories on rigid body motion using the modern mathematical formulation which is based on Lie Group Theory and Differential Topology (abstract manifolds). The goal of this research is to develop a mathematical model for kinematics and dynamics of a quadcopter, and the algorithms for trajectory control.

Motor Assumptions:

Standard motors used for quadcopter are brushless DC motors. For these types of motors, motor shaft is rigidly attached to the outer shell of the motor, and the motor axle is rigidly attached to the base of the motor, which is fixed on the quadcopter frame. When the motor is turned on, the axle of the motor is fixed relative to the quadcopter frame, and the outer shell is spinning around the axle.

Rotation of the motor will induce a torque on quadcopter frame. We assume that the magnitude of the generated torque is proportional to the rotational speed squared with a proportionality constant k_1.

$$\|\tau_i\| = k_1 {\dot{\varphi}_i}^2 \qquad (1).$$

The direction of this torque is always opposite to that the motor shell is spinning in. That is, if the motor shell is rotating counterclockwise, then the torque induced by this rotation is clockwise, acting on the quadcopter frame.

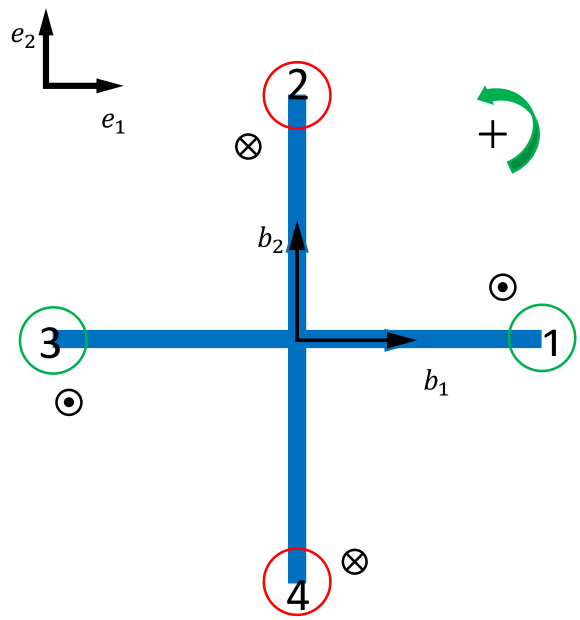

Figure 1.

Let's assume positive rotation for all motors to be counterclockwise. When all motors rotate in positive direction, rotors of motor 1 and 3 will generate upward thrust force (along positive b_3 direction), and motor 2 and 4 will generate downward thrust force (along negative b_3 direction). All thrust forces generated by four motors are parallel to body axis b_3 . In another words, the direction of thrust force generated by each rotor is always upright with respect to quadcopter frame regardless of the orientation of the body. For the quadcopter to fly properly, all rotors must generate downward thrust force, that is, motors 1 and 3 must rotate counterclockwise and motors 2 and 4 must rotate clockwise.

In addition, magnitude of the forces generated by rotors are proportional to each rotor's rotational speed squared:

$$||F_i|| = k_2 \dot{\varphi}_i{}^2 \qquad (2).$$

$\dot{\varphi}_i$ is the rotational speed of each motor, and k_2 is a proportionality constant.

Newtonian Setup:

Equations (1) and (2) given above only tell us what the magnitude of the force and torque is, and we need to somehow incorporate the direction of thrust force and motor torque into above expressions. Here we implement a little mathematical trick: rewrite$\dot{\varphi}_i{}^2$in above equations as $|\dot{\varphi}_i|\dot{\varphi}_i$, and in doing so allow us to determine the direction of the thrust force and motor torque given the sign of $\dot{\varphi}_i$ based on our sign convention. Now we rewrite equations (1) and (2) for all four motors:

$$F_1 = k_2|\dot{\varphi}_1|\dot{\varphi}_1\widehat{b_3} \quad (3)$$

$$F_2 = -k_2|\dot{\varphi}_2|\dot{\varphi}_2\widehat{b_3} \quad (4)$$

$$F_3 = k_2|\dot{\varphi}_3|\dot{\varphi}_3\widehat{b_3} \quad (5)$$

$$F_4 = -k_2|\dot{\varphi}_4|\dot{\varphi}_4\widehat{b_3} \quad (6)$$

$$\tau_1 = -k_1|\dot{\varphi}_1|\dot{\varphi}_1\widehat{b_3} \quad (7)$$

$$\tau_2 = -k_1|\dot{\varphi}_2|\dot{\varphi}_2\widehat{b_3} \quad (8)$$

$$\tau_3 = -k_1|\dot{\varphi}_3|\dot{\varphi}_3\widehat{b_3} \quad (9)$$

$$\tau_4 = -k_1|\dot{\varphi}_4|\dot{\varphi}_4\widehat{b_3} \quad (10)$$

The thrust forces will also induce another set of torques about the center of mass of the quadcopter, and they are determined by taking the cross product in $\mathbb{R}^3$ between the trust forces and the length vectors of each arm of the frame (which are the position vectors in the body frame, specifying the locations where thrust forces are applied). It is obvious these torques are also external torques to the system. Expressions for the induced torques are shown below, and we denote the length of each arm by l .

$$\tau_{F_1} = -lk_2|\dot{\varphi}_1|\dot{\varphi}_1\widehat{b_2} \quad (11)$$

$$\tau_{F_2} = l(-k_2|\dot{\varphi}_2|\dot{\varphi}_2)\widehat{b_1} \quad (12)$$

$$\tau_{F_3} = lk_2|\dot{\varphi}_3|\dot{\varphi}_3\widehat{b_2} \quad (13)$$

$$\tau_{F_4} = -l(-k_2|\dot{\varphi}_4|\dot{\varphi}_4)\widehat{b_1} \quad (14)$$

Our goal is to construct a mathematical model to describe the dynamics of the quadcopter, and in the Newtonian setup of this problem, we need to utilize conservation of linear and angular momentum, i.e. Newton-Euler equations on rigid bodies, to solve the problem.

$$\frac{dP_i}{dt} = F_{i,ex} \quad (15)$$

$$\frac{dL_i}{dt} = \tau_{i,ex} \quad (16).$$

In above equations, P_i and L_i denote linear and angular momentum of each rigid body in the system respectively, and $F_{i,ex}$, $\tau_{i,ex}$ denote external forces and torques applied to each rigid body respectively. According to our setup, there are 5 rigid bodies in the system: quadcopter frame and 4 identical motors. It is also easy to realize that eqns. (15) and (16) are second order non-linear differential equations, so for the quadcopter system, we will have a system of second order non-linear differential equations, and it is obvious that we do not want to solve this system of differential equations. So is there a better way to set up this problem to make the calculation easier? Fortunately for us there is, and next we will setup this problem in Lagrangian framework.

Lagrangian Setup:

First let us look at the configuration space of the system. To specify the position and orientation of the system, we need to know the position and orientation of each rigid body in the system. The configuration space of our system is thus $(\mathbb{R}^3 \times SO(3)) \times (\mathbb{R}^3 \times SO(3))^4$, where $(\mathbb{R}^3 \times SO(3))$ specifies the position and orientation of the frame, and $(\mathbb{R}^3 \times SO(3))^4$ specifies the position and orientation of 4 motors. We will explain the notion of $SO(3)$ in the next paragraph. For now, it is just a way of representing the orientation of a rigid body. It is also easy to realize that if given the position and orientation of the frame, then the position of all 4 motors are determined, thus the configuration manifold is reduced to $Q = (\mathbb{R}^3 \times SO(3)) \times (SO(3))^4$. Further, the orientation of each motor can be determined by only one parameter because all motors are fixed on the quadcopter frame and can only rotate about the motor axle, which means each motor has only one degree of freedom with respect to the copter frame that is the rotation about the motor axle. This further simplifies the configuration manifold of the system to $Q = (\mathbb{R}^3 \times SO(3)) \times (SO(2))^4$, which tells us the minimum number of parameters needed to characterize the configuration (position and orientation) of the entire system.

We further realize that we only have control over the rotational speed of each individual motors, and it is the rotation of these motors that makes the system move, so it is obvious that the control forces to the system are the thrust forces and motor torques. Since we are not interested in the configurations of the motors in this particular problem, we can replace all motors with thrust forces and motor torques that they generate. Now the system is simplified down to only the quadcopter frame, with external forces and torques.

As mentioned before, the orientation of a rigid body in space can be represented as an element in $SO(3)$. This is a choice of three mutually orthogonal unit vectors representing the forward/backward direction, the left/right direction, and the up/down direction for the rigid body; that is a frame for the body. We can describe an element in $SO(3)$ as a matrix $R = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$ where b_1, b_2, b_3 are the mutually orthogonal vectors describing the orientation of the body relative to the inertial frame $e_1 \quad e_2 \quad e_3$. The matrix R is then considered to be the element in $SO(3)$. The matrix R is an 3×3 orthogonal matrix, meaning $R^T R = I$ or $R^{-1} = R^T$; this reflects the $'O'$ in $SO(3)$. Further, $\det(R) = 1$ which means it is special, the$'S'$ in $SO(3)$, among all orthogonal matrices. We will later need that $SO(3)$ forms a group under matrix multiplication that $A, B \in SO(3)$ implies $AB \in SO(3)$.

A very useful property in describing rotation is that if we consider $R = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$ as a time dependent function we have:

$$\dot{b_1} = -\beta b_3 + \alpha b_2 \tag{17}$$

$$\dot{b_2} = \theta b_3 - \alpha b_1 \tag{18}$$

$$\dot{b_3} = -\theta b_2 + \beta b_1 \qquad (19),$$

where α, β, θ are rates of change. With this property of $\dot{R}$, we can write the angular velocity of the frame given by definition

$$\hat{\Omega} = R^T \cdot \dot{R} \tag{20}$$

$\hat{\Omega}$ is the angular velocity matrix, in term of $\alpha, \beta, and\ \theta$, $\hat{\Omega}$ can be written explicitly as

$$\hat{\Omega} = \begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\theta \\ -\beta & \theta & 0 \end{bmatrix} \tag{21}$$

From the skew-symmetric nature of angular velocities, we have angular velocity vector as

$$\Omega = < \theta, \beta, \alpha > \ \in \mathbb{R}^3 \tag{22}$$

The transformation from angular velocity matrix to angular velocity vector is given by the map

$$\hat{\cdot} : \ \mathbb{R}^3 \longrightarrow \mathbb{R}^{3\times3} \qquad (23),$$

this implies

$$u \times v = \hat{u} \cdot v \qquad (24),$$

where $u, v \in \mathbb{R}^3$are vectors of $\mathbb{R}^3$, and "×" is the standard cross product in $\mathbb{R}^3$, and "·" is the matrix multiplication. In order for the matrix multiplication to work, vectors u, v need to be written in column form. From now on, unless stated explicitly, single letter denote a vector and single letter with a "hat" on top denotes the skew-symmetric matrix form of that vector.

The linear velocity of the center of mass of the quadcopter frame is given by $v = < \dot{x}, \dot{y}, \dot{z} >$, where x, y, z are the coordinates of the center of mass of the frame in inertia frame $[e_1 \quad e_2 \quad e_3]$.

Before proceeding further, we shall introduce the notion of the tangent bundle of a manifold and the Riemannian metric. By definition, the collection of all tangent vectors at a point x in the manifold M is the tangent space at that point, denoted by T_xM. The disjoint union of all tangent spaces on the manifold is called the tangent bundle, denoted by TM,

$$TM = \cup_{x \in M} T_xM \tag{25}$$

Locally, for a m-dimensional manifold, tangent bundle TM can be viewed as $u \times R^m$, where $u \subset M$, and $T_xM \cong R^m$. The tangent bundle is equipped with a natural projection map π_{TM} which maps an element in the total space TM to an element in the base space M $\pi_{TM}: TM \to M$, defined by

$$\pi_{TM}(v) = x, \qquad \text{when } v \in T_xM \tag{26}$$

The Riemannian metric on a manifold denoted by $g(,)$ is a positive definite bilinear map that takes two nonzero tangent vectors at a point and returns a nonzero positive real number $g: T_xM \times T_xM \to \mathbb{R}$, where $x \in M$. The Riemannian metric is a way of measuring "distance", and is in a sense the "inner-product" on a manifold.

With the ideas of tangent spaces and metric in mind, we realize that linear and angular velocities are tangent vectors to the configuration manifold, and kinetic energy of the system produces a metric-the kinetic energy metric-that measures the "distance"/energy on the configuration manifold.

By definition, the kinetic energy of the system is given by

$$KE = KE_{tran} + KE_{rot} \tag{27},$$

where KE_{tran} is the total translational kinetic energy and KE_{rot} is total rotational kinetic energy:

$$KE_{tran} = \frac{1}{2} M \|v\|_{\mathbb{R}^3}{}^2 \tag{28}$$

$$KE_{rot} = \frac{1}{2} G_{\mathbb{R}^3}(\mathrm{I}_c(\Omega), \Omega) \tag{29}.$$

In above eqns. (28) and (29), "$G_{\mathbb{R}^3}(\,,)$" denotes the standard inner product in $\mathbb{R}^3$, and $\mathrm{I}_c(\Omega)$ is the moment of inertia tensor $\mathrm{I}_c(\,)$ applied to the angular velocity vector Ω,

$$\mathrm{I}_c(\Omega) = \int_V (X_i - X_c) \times (\Omega \times (X_i - X_c))\rho \cdot dv \tag{30}.$$

The kinetic energy for the system is calculated explicitly to be

$$KE = \begin{bmatrix} \begin{matrix} \frac{1}{2}M\dot{x}^2 & 0 & 0 \\ 0 & \frac{1}{2}M\dot{y}^2 & 0 \\ 0 & 0 & \frac{1}{2}M\dot{z}^2 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \frac{1}{2}I_{11}\theta^2 & 0 & 0 \\ 0 & \frac{1}{2}I_{22}\beta^2 & 0 \\ 0 & 0 & \frac{1}{2}I_{33}\alpha^2 \end{matrix} \end{bmatrix} \tag{31},$$

where M is the mass of the quadcopter frame and I_{11}, I_{22}, I_{33} are moment of inertia of the frame with respect to body axis b_1, b_2, b_3 respectively.

Using kinetic energy of the system, we can calculate the kinetic energy metric for the manifold by

$$G_{ij} = \frac{\partial^2 KE}{\partial v^i \partial v^j} \tag{32},$$

where v^i, v^j are first derivative of the free parameters on the manifold (i.e. $\dot{x}, \dot{y}, \dot{z}, \alpha, \beta, \theta$).

Carrying out the calculation we obtain the kinetic energy metric:

$$G = \begin{bmatrix} M & 0 & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{33} \end{bmatrix} \tag{33},$$

where again M is the mass of the quadcopter frame and I_{11}, I_{22}, I_{33} are moment of inertia of the frame with respect to body axis b_1, b_2, b_3 respectively. Next, we calculate the Christoffel symbols for this metric

$$\Gamma_{ij}^k = \frac{1}{2} G^{kl} \left(\frac{\partial G_{li}}{\partial x^j} + \frac{\partial G_{jl}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^l} \right) \tag{34}.$$

Noticing that all nonzero entries in the metric are constants, we conclude that all Christoffel symbols are zero, so we have a flat manifold.

Recall that we have already set up the external forces and torques in Newtonian framework, what we need to do now is to transfer that information into Lagrangian framework, i.e. representing forces and torques on configuration manifold. For each element in the tangent bundle of the configuration manifold, $v_q \in TQ$, a Lagrangian force, $F_{(t,v_q)}$, is a dual vector in the dual tangent space of the configuration manifold, $F_{(t,v_q)} \in T_q^*Q$, such that,

$$F_{(t,v_q)}(w_q) = G_{\mathbb{R}^3}\big(f(t,v_q),\, V(w_q)\big) + G_{\mathbb{R}^3}\big(\tau(t,v_q),\, \Omega(w_q)\big) \tag{35}$$

for all $w_q \in T_qQ$. $V(w_q)$, and $\Omega(w_q)$ return linear and angular velocities respectively. Lagrangian force $F_{(t,v_q)}$ is a way of modeling the effect of Newtonian forces (f/τ) in Lagrangian mechanics. Since $F_{(t,v_q)}$ is a dual vector, it can be written as

$$F = F_x\, dx + F_y\, dy + F_z\, dz + F_\theta\, d\theta + F_\beta\, d\beta + F_\alpha\, d\alpha \tag{36}$$

Using eqns. (3) through (13), and (35), $F_{(t,v_q)}$ can be written component-wise as follow:

$$F_x = k_2 G_{\mathbb{R}^3}\big(\widehat{b_3}, \widehat{e_1}\big)(|\dot\varphi_1|\dot\varphi_1 - |\dot\varphi_2|\dot\varphi_2 + |\dot\varphi_3|\dot\varphi_3 - |\dot\varphi_4|\dot\varphi_4) \tag{37}$$

$$F_y = k_2 G_{\mathbb{R}^3}\big(\widehat{b_3}, \widehat{e_2}\big)(|\dot\varphi_1|\dot\varphi_1 - |\dot\varphi_2|\dot\varphi_2 + |\dot\varphi_3|\dot\varphi_3 - |\dot\varphi_4|\dot\varphi_4) \tag{38}$$

$$F_z = k_2 G_{\mathbb{R}^3}\big(\widehat{b_3}, \widehat{e_3}\big)(|\dot\varphi_1|\dot\varphi_1 - |\dot\varphi_2|\dot\varphi_2 + |\dot\varphi_3|\dot\varphi_3 - |\dot\varphi_4|\dot\varphi_4) - Mg \tag{39}$$

$$F_\theta = lk_2(|\dot\varphi_4|\dot\varphi_4 - |\dot\varphi_2|\dot\varphi_2) \tag{40}$$

$$F_\beta = lk_2(|\dot\varphi_3|\dot\varphi_3 - |\dot\varphi_1|\dot\varphi_1) \tag{41}$$

$$F_\alpha = -k_1(|\dot\varphi_1|\dot\varphi_1 + |\dot\varphi_2|\dot\varphi_2 + |\dot\varphi_3|\dot\varphi_3 + |\dot\varphi_4|\dot\varphi_4) \tag{42}$$

Now we have all the setups necessary to write down the equations of motion governing the dynamics of the quadcopter, using Lagrange-d 'Alembert principles on Riemannian manifolds, which is given by

$$\nabla_{\gamma'(t)}\gamma'(t) = G^\#(F_{t,\gamma'(t)}) \tag{43}$$

where $\gamma(t)$ is a curve on the configuration manifold, and the derivative of $\gamma(t)$, $\gamma'(t)$, is in the tangent space of the manifold at $\gamma(t), \gamma'(t) \in T_{\gamma(t)}Q$. Terms on the left-hand side of the equation, $\nabla_{\gamma'(t)}\gamma'(t)$, is the covariant derivative of the tangent vector field $\gamma'(t)$ with respect to itself. On the right-hand side of the equation, $G^\#(F_{t,\gamma'(t)})$ is the "$G\ sharp$" map applied to the Lagrangian force. "$G\ sharp$" map takes in a dual vector and returns a vector associated to this dual vector through the metric on the manifold:

$$G^\#: v^* \to v,\ v^* \in T_q^*Q\,, v \in T_qQ \tag{44}$$

such that $v^*(w) = G(v,w)$, for all $w \in T_qQ$, that is dual vector v^* acting on any vector w on the manifold will return the same answer as when metric tensor G acts on vectors $v\ and\ w$, where v is the image of the "sharp" map. In coordinates, $\nabla_{\gamma'(t)}\gamma'(t)$ can be written explicitly as

$$\nabla_{\gamma'(t)}\gamma'(t) = (\ddot\gamma^k(t) + \Gamma^k_{ij}\dot\gamma^i(t)\dot\gamma^j(t))\frac{\partial}{\partial x^k} \tag{45}$$

and $G^\#(F_{t,\gamma'(t)})$ can be written out as

$$G^\#\big(F_{t,\gamma'(t)}\big) = F_j g^{ij}\frac{\partial}{\partial x^i} \tag{46}$$

where g^{ij} is the inverse metric. Write out Lagrange-d' Alembert equations in component form, we get:

$$\ddot{x} = \frac{1}{M} k_2 G_{\mathbb{R}^3}\left(\widehat{b_3}, \widehat{e_1}\right)(|\dot{\varphi}_1|\dot{\varphi}_1 - |\dot{\varphi}_2|\dot{\varphi}_2 + |\dot{\varphi}_3|\dot{\varphi}_3 - |\dot{\varphi}_4|\dot{\varphi}_4) \qquad (47)$$

$$\ddot{y} = \frac{1}{M} k_2 G_{\mathbb{R}^3}\left(\widehat{b_3}, \widehat{e_2}\right)(|\dot{\varphi}_1|\dot{\varphi}_1 - |\dot{\varphi}_2|\dot{\varphi}_2 + |\dot{\varphi}_3|\dot{\varphi}_3 - |\dot{\varphi}_4|\dot{\varphi}_4) \qquad (48)$$

$$\ddot{z} = \frac{1}{M} (k_2 G_{\mathbb{R}^3}\left(\widehat{b_3}, \widehat{e_3}\right)(|\dot{\varphi}_1|\dot{\varphi}_1 - |\dot{\varphi}_2|\dot{\varphi}_2 + |\dot{\varphi}_3|\dot{\varphi}_3 - |\dot{\varphi}_4|\dot{\varphi}_4) - Mg) \qquad (49)$$

$$\dot{\theta} = \frac{1}{I_{11}} l k_2 (|\dot{\varphi}_4|\dot{\varphi}_4 - |\dot{\varphi}_2|\dot{\varphi}_2) \qquad (50)$$

$$\dot{\beta} = \frac{1}{I_{22}} l k_2 (|\dot{\varphi}_3|\dot{\varphi}_3 - |\dot{\varphi}_1|\dot{\varphi}_1) \qquad (51)$$

$$\dot{\alpha} = -\frac{1}{I_{33}} k_1 (|\dot{\varphi}_1|\dot{\varphi}_1 + |\dot{\varphi}_2|\dot{\varphi}_2 + |\dot{\varphi}_3|\dot{\varphi}_3 + |\dot{\varphi}_4|\dot{\varphi}_4) \qquad (52).$$

Equations (47), (48), and (49) can be easily solved by numerical integration, however, it is required to know the orientation of the quadcopter frame to calculate $G_{\mathbb{R}^3}(\hat{b}_i, \hat{e}_j)$. To get the orientation of the frame as a function of time, we need to solve eqns. (50), (51), and (52) first. Using numerical integration, we can get rates of change α, β, θ as functions of time, but these are not the orientations of the frame. Take one step back, as we mentioned before, orientations of the quadcopter can be represented by elements of $SO(3)$, and $SO(3)$ is a matrix Lie group with respect to operation of matrix multiplication. Lie group is a group that is also a manifold with a smooth inverse group operation. We then realize that rates of change of the orientation $< \theta, \beta, \alpha > \in \mathbb{R}^3$, and $\mathbb{R}^3$ is isomorphic to vector space $\mathfrak{so}(3)$, the space of all 3×3 skew-symmetric matrices, through $\hat{\cdot}$ map (see eqn. (23)).

It is a known fact that $\mathfrak{so}(3)$ is the Lie algebra of the matrix Lie group $SO(3)$, since we can solve for the elements in $\mathfrak{so}(3)$, it would be nice if there is a way to get elements of $SO(3)$ through elements of $\mathfrak{so}(3)$, and indeed, there is such a map: the exponential map.

$$exp\colon\ \mathfrak{g} \to G \qquad (53).$$

$\mathfrak{g}$ is the Lie algebra, and G is the corresponding Lie group. One of the remarkable properties of exponential map tells us that if the Lie group is a matrix Lie group, which it is in our case, then the exponential map is the matrix exponential map. The matrix exponential is defined by $exp\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ as

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \qquad A \in \mathbb{R}^{n\times n} \qquad (54).$$

In our case, the matrices As are elements in the Lie algebra that is they are 3×3 skew symmetric matrices, thus we can simplify the matrix exponential map using the Rodrigues' Formula for skew symmetric matrices:

For $w \in \mathbb{R}^3 \quad \widehat{w} \in \mathfrak{so}(3)$,

$$\exp(\widehat{\omega}) = \begin{cases} I_3 & , \quad \omega = 0 \\ I_3 + \frac{sin\|\omega\|_{\mathbb{R}^3}}{\|\omega\|_{\mathbb{R}^3}} \widehat{\omega} + \frac{1 - cos\|\omega\|_{\mathbb{R}^3}}{\|\omega\|_{\mathbb{R}^3}{}^2} \widehat{\omega}^2 & , \quad \omega \neq 0 \end{cases} \qquad (55)$$

$\exp(\widehat{\omega}) \in SO(3)$.

To understand how the matrix exponential map helps us find the orientation of the quadcopter, we first need to introduce the concept of left invariant map and left invariant vector fields on the Lie group G.

Left invariant map L_g:

$$L_g: h \to g * h, \quad g, h \in G, \text{ for all } h, \text{ and “*” is group action} \quad (56)$$

Left invariant vector fields:

$$\varepsilon_L(g) = T_e L_g(\varepsilon_L(e)) \quad (57),$$

where $\varepsilon_L(e)$ is the left invariant vector field at the identity, and $T_e L_g$ is the derivative map of the left invariant map:

$$T_e L_g: T_e G \to T_{g*e} G = T_g G \quad (58).$$

It is obvious that the left invariant vector fields on a Lie group is determined by the tangent vectors at the identity, so the set of left-invariant vector fields denoted by $\mathfrak{L}(G)$ is isomorphic to the tangent space at the identity $T_e G$. It is also known that the Lie algebra $\mathfrak{g}$ of a lie group G is the tangent space at the identity $T_e G$, so the Lie algebra of the Lie group defines the set of left invariant vector fields on the lie group.

Now let's return to the quantity $\exp(\varepsilon)$, given by the properties of the exponential map on a Lie group:

$$exp(\varepsilon) = \Phi_1^{\varepsilon_L}(e), \qquad \varepsilon \in T_e G, \varepsilon_L \in \mathfrak{L}(G) \quad (59).$$

ε_L is related to ε by $\varepsilon_L(e) = \varepsilon$. $\Phi^{\varepsilon_L}(e)$ is the integral curve $\gamma(t)$: $\mathbb{R} \to G$ of the left invariant vector field starting at the identity $\Phi_1^{\varepsilon_L}(e)$ is the value of the curve at t = 1. From above, we deduce that

$$exp(\varepsilon t) = \Phi_t^{\varepsilon_L}(e) \quad (60).$$

physically, it means if the orientation of our rigid body, starting out at the identity, is changing at a constant rate given by the vector ε, then after time period t, the orientation of the rigid body will be the value of the integral curve at time equals to t. If the orientation of our frame did not start out at the identity, we can use the left invariant map to translate the final orientation to that starting orientation.

$$\Phi_t^{\varepsilon_L}(g) = L_g(exp(\varepsilon t)) \quad (61).$$

In our case, the Lie group is a matrix lie group,

$$L_g(exp(\varepsilon t)) = g \cdot \exp(\varepsilon t) \quad (62),$$

Left invariant map is just the left matrix multiplication. We have justified the use of matrix exponential map to get the orientation of the quadcopter given the rate of change of the orientation as a function of time:

$$R_{t+\Delta t} = R_t \cdot \exp(\widehat{\omega} \cdot \Delta t) \quad (63).$$

After solving eqns. (50), (51), and (52) numerically using improved Newton's method, we obtain the rates of change of orientation of the quadcopter frame as functions of time, and using that information, we can solve eqn. (63) assuming the rate of change of orientation is constant for a very small time interval. We then back-substitute answers acquired from solving eqn. (63) into eqns. (47), (48), and (49) to get the position of the center of mass of the copter frame. The position and orientation of the frame together completely describe the configuration of the quadcopter frame in space at any point in time.

Simulation:

Equations of motion obtained by solving eqns. (47)-(52) specify the movement of the quadcopter frame given any control inputs, the $\dot{\varphi}_i$s; this nature of the equations of motion allows us to create flight patterns by controlling the motor rotational speeds. Theoretically speaking, any possible configuration of the copter frame can be achieved by inputting appropriate motor speeds into the system. However, in actuality, one often runs into problems such that the final or goal configuration of the frame is known, and one needs to back out the control inputs. This type of problems are commonly referred to as inverse kinematics problems, and these problems are often algebraically complex, and close form solutions are usually not obtainable. In our case, we solve the inverse kinematics problems simply by trial and error. We make educated guesses to what input functions could look like to achieve desired final configuration and adjust them based on the difference between simulation output and target configuration. After obtaining the correct inputs, we run simulation again to visualize the movement of the frame using these inputs. For the purpose of this research, we simulate only two simple fight patterns: the square flight pattern and the takeoff moving forward flight pattern. MATLAB scripts attached in the appendix are simulations of these two flight patterns with correct inputs. We recognize that although the motor speed control functions we cooked up do yield desired results, these functions are not necessary the "best" solutions. In the future, we hope to develop a systematic method to solve these inverse kinematics problems instead of "guessing and checking".

Conclusion and Future Work:

Here we conclude that the dynamics model of the quadcopter frame, subject to external forces and torques modeled according to our assumptions, is given by eqns. (47)-(52), and eqn. (63). In another words, these equations completely describe the dynamics and kinematics of the quadcopter frame. Numerical solutions to these equations are obtained using MATLAB, details can be found in annotated MATLAB script files in the Appendix. Using our model, we also create different flight patterns using MATLAB to demonstrate the validity of our model. In the Appendix, we include MATLAB functions and scripts for a square shaped flight pattern, and a simple take off moving forward flight pattern. Again, details of these simulations can be found in the annotation of corresponding script files in the Appendix.

The main focus of our work is the dynamics modeling of the quadcopter; with that being said, we haven't touched anything related to the control of this mechanical system, and the implementation of control theory will be the next phase of this project. Although we implemented flight control algorithms for the purpose of simulation, these algorithms are merely open loop commands, and our algorithms will not respond correctly to any form of external disturbances nor will they function properly when the modeling assumptions are not met. In a sense, we basically "hard-coded" these flight controls just for the sake of demonstration. In order to achieve more advanced and natural flight maneuvers, feed-back controls must be implemented to our existing dynamics model. In addition to the subject of control theory, one might also want to improve the modeling assumptions, especially in the area of external forces and torques. Our assumptions on forces and torques are very rudimentary, pretty much the simplest form of aerodynamic forces generated by rotors, and we also ignored other external effects independent of rotor aerodynamics, like the effect of wind. Further refinement of these assumptions will certainly improve our modeling accuracy.