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Flattening a Cone

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Contents

1 How this problem got started

A local manufacturing design company called with the following problem.

> We want to manufacture a cut off slanted cone from a flat sheet of metal. If the cone was a normal right cone we know that we would simply cut out a sector of a circle and roll it up. However the cone is slanted. We want to know what the flattened shape looks like so that we can cut it out and roll it up to closely approximate correct final shape. We also want to minimize the amount of wasted metal after the shape is cut out.

I replied that

> No, I don't know of any formula for this, let me think about it. Can you send me a picture.

The company sent a CAD drawing. The drawing is the first of the pictures in the pictures and attachment section (Section 5). The problem, and it generalizations may be solved analytically but the analytical solution is given in terms of indefinite integrals which rarely can be evaluated in closed form. The solutions may be found numerically which are good enough the create a picture of the flattened out cone. In Section 2 we describe the problem of flattening out a cone over a curve as a parametrization problem in the differential geometry of surfaces. In Section 3 we verify that the cone surface really may be flattened out by showing that the flattening map (or is inverse the roll up map) is an isometry. In Section 4 we carry out the computations in the motivating example, coming up with formulas to describe the outline of the flattened out shape. Finally Section 5 shows pictures of the rolled up cone, the flattened out region, and the Maple worksheet that computes the flattened out region. The worksheet also contains several views of the cones to supply a visual verification.

2 Problem statement and solution

Let $P(s), 0 \leq s \leq L$, be space curve of length L parameterized by arclength s and let Q be any point in $\mathbb{R}^3$. Consider the cone over Q determined by $P(s)$, we give the detailed definition shortly. The cone is flat in the differential geometry sense and therefore may be flattened to a region S in the plane. The goal of this note is to describe the region S in terms of polar coordinates. The solution to this problem is important in manufacturing where the flattened region is rolled into a cone or a truncated cone shape. An important function in our analysis will be distance from Q to $P(s)$

$$\ell(s) = \|P(s) - Q\| \, . \tag{1}$$

Define the cone map A from the rectangle $R = [0, L] \times [0, 1]$ by

$$A(s, t) = (1 - t)Q + tP(s)$$

The image C of A, is called the cone over $P(s)$ based at Q. We would like to flatten C into a sector S. Namely,

- find a sector S in the plane described in polar coordinates by

$$S = \{(r, \theta) : 0 \leq \theta \leq \Theta, 0 \leq r \leq \rho(\theta)\} \tag{2}$$

  for some Θ and $\rho(\theta)$ to be determined, and

- a map $B : S \to R$

$$B(r, \theta) = \left(\sigma(\theta), \frac{r}{\rho(\theta)}\right)$$

  for σ to be determined,

- such that the composite map $A \circ B$ given by

$$AB(r, \theta) = \left(1 - \frac{r}{\rho(\theta)}\right) Q + \left(\frac{r}{\rho(\theta)}\right) P(\sigma(\theta)) \tag{3}$$

  is a local isometry from S onto C, except at the cone point.

In the composite map $A \circ B$ the origin is mapped to the cone point Q,the curve given $r = \rho(\theta)$ is mapped to the curve $P(s)$, and the radial line segment $0 \le r \le \rho(\theta)$ is mapped isometrically to a segment determined by Q and a point on the path. Hence $AB(r(\theta), \theta) = P(\sigma(\theta))$, for some function σ and the complete formula for the map AB is completely determined from the geometry. If the map of a radial line segment is to be isometric then we must have

$$\rho(\theta) = \ell(\sigma(\theta)), \tag{4}$$

thus $\rho(\theta)$ is determined once $\sigma(\theta)$ is known. Since the curve $r = \rho(\theta)$ is mapped isometrically to the curve $AB(r(\theta), \theta) = P(\sigma(\theta))$ Then we must have:

$$\begin{aligned}
\int_0^\theta \left\| \frac{d}{du} P(\sigma(u)) \right\| du &= \int_0^\theta \sqrt{(\rho(u))^2 + (\rho'(u))^2}\, du \\
\int_0^\theta |\sigma'(u)| \, \|P'(\sigma(u))\| \, du &= \int_0^\theta \sqrt{(\rho(u))^2 + (\rho'(u))^2}\, du \\
\int_0^\theta |\sigma'(u)| \, du &= \int_0^\theta \sqrt{(\rho(u))^2 + (\rho'(u))^2}\, du
\end{aligned}$$

since the left and right hand sides are the arclengths in three space and polar coordinates. It follows that $\sigma(\theta)$ is simple arclength along the curve $r = \rho(\theta)$ and

$$(\sigma'(\theta))^2 = (\rho(\theta))^2 + (\rho'(\theta))^2 \tag{5}$$

From this equation and equation 4 we get

$$\begin{aligned}
(\sigma'(\theta))^2 &= (\rho(\theta))^2 + (\rho'(\theta))^2 \\
&= (\ell(\sigma(\theta)))^2 + (\ell'(\sigma(\theta))\sigma'(\theta))^2 \\
(1 - (\ell'(\sigma(\theta)))^2)\,(\sigma'(\theta))^2 &= (\ell(\sigma(\theta)))^2 \\
(\sigma'(\theta))^2 &= \frac{(\ell(\sigma(\theta)))^2}{(1 - (\ell'(\sigma(\theta)))^2)} \\
\sigma'(\theta) &= \pm \frac{\ell(\sigma(\theta))}{\sqrt{(1 - (\ell'(\sigma(\theta)))^2)}}
\end{aligned}$$

Simplifying by setting $s = \sigma(\theta)$, $\frac{ds}{d\theta} = \sigma'(\theta)$ we get the differential equation

$$\frac{ds}{d\theta} = \frac{\ell(s)}{\sqrt{1 - (\ell'(s))^2}}.$$

This differential equation can be solved by separating and integrating

$$\theta = \int_0^{\sigma(\theta)} \frac{\sqrt{1 - (\ell'(s))^2}}{\ell(s)}\, ds \tag{6}$$

In particular

$$\Theta = \int_0^L \frac{\sqrt{1-(\ell'(s))^2}}{\ell(s)} ds \tag{7}$$

Defining $T(\sigma)$ by

$$T(\sigma) = \int_0^\sigma \frac{\sqrt{1-(\ell'(s))^2}}{\ell(s)} ds \tag{8}$$

we may define σ by

$$\sigma(\theta) = T^{-1}(\theta) \tag{9}$$

Proposition 1 *Let $P(s), \ell(s), T(\sigma), \sigma(\theta), \rho(\theta) = \ell(\sigma(\theta)), A, B, \Theta$ be as defined above. Then the sector S , which is the domain of the complete cone map*

$$AB(r,\theta) = \left(1 - \frac{r}{\rho(\theta)}\right) Q + \left(\frac{r}{\rho(\theta)}\right) P(\sigma(\theta)), \tag{10}$$

is defined by

$$S = \{(r,\theta) : 0 \le \theta \le \Theta, 0 \le r \le \rho(\theta)\} \tag{11}$$

Example 2 *Let $P(s) = (a\cos\left(\frac{s}{a}\right), a\sin(\frac{s}{a}), c), 0 \le s \le 2\pi a$, and $Q = 0$ then $\ell(s) = \sqrt{a^2+c^2}$ and*

$$\theta = \int_0^{\sigma(\theta)} \frac{ds}{\sqrt{a^2+c^2}}$$

$$\theta = \frac{\sigma(\theta)}{\sqrt{a^2+c^2}}$$

$$\sigma(\theta) = \theta\sqrt{a^2+c^2}$$

Since $0 \le s \le 2\pi a$ then

$$0 \le \sigma(\theta) \le 2\pi a$$

$$0 \le \theta\sqrt{a^2+c^2} \le 2\pi a$$

$$0 \le \theta \le \frac{2\pi a}{\sqrt{a^2+c^2}}$$

and so

$$\Theta = \frac{2\pi a}{\sqrt{a^2+c^2}}$$

$$\rho(\theta) = \sqrt{a^2+c^2}$$

3 Verification that AB is a local isometry

We need to show that a orthonormal frame on S is taken to an orthonormal frame on C. On S we take the orthonormal frame $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$. We compute

$$\begin{aligned} dAB\left(\frac{\partial}{\partial r}\right) &= \frac{d}{dt}_{|t=0}\left(\left(1-\frac{r}{\rho(\theta)}\right)Q+\left(\frac{r}{\rho(\theta)}\right)P(\sigma(\theta))\right) \\ &= \frac{P(\sigma(\theta))-Q}{\rho(\theta)} \\ &= \frac{P(\sigma(\theta))-Q}{\ell(\sigma(\theta))} \end{aligned}$$

$$\begin{aligned} dAB\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) &= \frac{1}{r}\frac{d}{dt}_{|t=0}\left(\left(1-\frac{r}{\rho(\theta+t)}\right)Q+\left(\frac{r}{\rho(\theta+t)}\right)P(\sigma(\theta+t))\right) \\ &= \frac{1}{r}\left(\frac{r\rho'(\theta)}{\rho^2(\theta)}Q\right)+\frac{1}{r}\left(\frac{-r\rho'(\theta)}{\rho^2(\theta)}P(\sigma(\theta))+\frac{r\sigma'(\theta)}{\rho(\theta)}P'(\sigma(\theta))\right) \\ &= \frac{\rho'(\theta)}{\rho^2(\theta)}Q-\frac{\rho'(\theta)}{\rho^2(\theta)}P(\sigma(\theta))+\frac{\sigma'(\theta)}{\rho(\theta)}P'(\sigma(\theta)) \\ &= -\frac{\rho'(\theta)}{\rho^2(\theta)}\left(P(\sigma(\theta))-Q\right)+\frac{\sigma'(\theta)}{\rho(\theta)}P'(\sigma(\theta)) \end{aligned}$$

We calculate

$$dAB\left(\frac{\partial}{\partial r}\right)\bullet dAB\left(\frac{\partial}{\partial r}\right) = \left\|\frac{P(\sigma(\theta))-Q}{\ell(\sigma(\theta))}\right\| = 1$$

$$\begin{aligned} dAB\left(\frac{\partial}{\partial r}\right)\bullet dAB\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) &= \frac{P(\sigma(\theta))-Q}{\rho(\theta)}\bullet\left(-\frac{\rho'(\theta)}{\rho^2(\theta)}\left(P(\sigma(\theta))-Q\right)+\frac{\sigma'(\theta)}{\rho(\theta)}P'(\sigma(\theta))\right) \\ &= -\frac{\rho'(\theta)}{\rho(\theta)}\left\|\frac{P(\sigma(\theta))-Q}{\rho(\theta)}\right\|^2+\frac{\sigma'(\theta)}{\rho(\theta)^2}\left(P(\sigma(\theta))-Q\right)\bullet P'(\sigma(\theta)) \\ &= -\frac{\rho'(\theta)}{\rho(\theta)}+\frac{\sigma'(\theta)}{\rho(\theta)^2}\left(P(\sigma(\theta))-Q\right)\bullet P'(\sigma(\theta)) \end{aligned}$$

Now

$$\left(P(\sigma(\theta)))-Q\right)\bullet\left(P(\sigma(\theta)))-Q\right)=\rho^2(\theta)$$

and

$$\begin{aligned} 2\left(P(\sigma(\theta)))-Q\right)\bullet P'(\sigma(\theta))\sigma'(\theta) &= 2\rho'(\theta)\rho(\theta) \\ \left(P(\sigma(\theta)))-Q\right)\bullet P'(\sigma(\theta)) &= \frac{\rho'(\theta)\rho(\theta)}{\sigma'(\theta)} \end{aligned}$$

Continuing

$$
\begin{aligned}
dAB\left(\frac{\partial}{\partial r}\right) \bullet dAB\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) &= \frac{\rho'(\theta)}{\rho^2(\theta)} + \frac{\sigma'(\theta)}{\rho(\theta)^2}\left(P(\sigma(\theta)) - Q\right) \bullet P'(\sigma(\theta)) \\
&= -\frac{\rho'(\theta)}{\rho(\theta)} + \frac{\sigma'(\theta)}{\rho^2(\theta)}\frac{\rho'(\theta)\rho(\theta)}{\sigma'(\theta)} \\
&= 0
\end{aligned}
$$

Finally,

$$
\begin{aligned}
dAB\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) \bullet dAB\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) &= \left(-\frac{\rho'(\theta)}{\rho^2(\theta)}\left(P(\sigma(\theta)) - Q\right) + \frac{\sigma'(\theta)}{\rho(\theta)}P'(\sigma(\theta))\right) \bullet \\
&\left(-\frac{\rho'(\theta)}{\rho^2(\theta)}\left(P(\sigma(\theta)) - Q\right) + \frac{\sigma'(\theta)}{\rho(\theta)}P'(\sigma(\theta))\right) \\
&= \left(\frac{\rho'(\theta)}{\rho(\theta)}\right)^2 \left\|\frac{P(\sigma(\theta)) - Q}{\rho(\theta)}\right\|^2 + \left(\frac{\sigma'(\theta)}{\rho(\theta)}\right)^2 \|P'(\sigma(\theta))\|^2 \\
&- 2\frac{\rho'(\theta)}{\rho^2(\theta)}\frac{\sigma'(\theta)}{\rho(\theta)}\left(P(\sigma(\theta)) - Q\right) \bullet P'(\sigma(\theta)) \\
&= \left(\frac{\rho'(\theta)}{\rho(\theta)}\right)^2 + \left(\frac{\sigma'(\theta)}{\rho(\theta)}\right)^2 - 2\frac{\rho'(\theta)}{\rho^2(\theta)}\frac{\sigma'(\theta)}{\rho(\theta)}\frac{\rho'(\theta)\rho(\theta)}{\sigma'(\theta)} \\
&= \left(\frac{\rho'(\theta)}{\rho(\theta)}\right)^2 + \left(\frac{\sigma'(\theta)}{\rho(\theta)}\right)^2 - 2\left(\frac{\rho'(\theta)}{\rho(\theta)}\right)^2 \\
&= \frac{(\sigma'(\theta))^2 - (\rho'(\theta))^2}{(\rho(\theta))^2}
\end{aligned}
$$

By equation 5 this quantity equals 1.

4 The real example

Let the path be a circle in the plane that is offset from the origin, say radius a and center $(b, 0, 0)$, and let Q be the point $(0, 0, c)$ on the z-axis .We may assume that

$$
P(s) = (b + (a\cos\left(\frac{s}{a}\right), a\sin(\frac{s}{a}), 0)
$$

Then

$$
\begin{aligned}
\ell(s) &= \sqrt{a^2 + b^2 + c^2 + 2ab\cos\left(\frac{s}{a}\right)} \\
\ell'(s) &= \sqrt{a^2 + b^2 + c^2 + 2ab\cos\left(\frac{s}{a}\right)} \\
&= \frac{-b\sin\frac{s}{a}}{\sqrt{a^2 + b^2 + c^2 + 2ab\cos\left(\frac{s}{a}\right)}}
\end{aligned}
$$

So that

$$\begin{aligned}
\frac{\sqrt{1-(\ell'(s))^2}}{\ell(s)} &= \frac{\sqrt{1-\frac{b^2\sin^2\frac{s}{a}}{a^2+b^2+c^2+2ab\cos\left(\frac{s}{a}\right)}}}{\sqrt{a^2+b^2+c^2+2ab\cos\left(\frac{s}{a}\right)}} \\
&= \sqrt{\frac{a^2+c^2+2ab\cos\frac{s}{a}+b^2\cos^2\frac{s}{a}}{a^2+b^2+c^2+2ab\cos\frac{s}{a}}} \\
&= \frac{\sqrt{a^2+c^2+2ab\cos\frac{s}{a}+b^2\cos^2\frac{s}{a}}}{a^2+b^2+c^2+2ab\cos\frac{s}{a}}
\end{aligned}$$

And

$$\begin{aligned}
T(\sigma) &= \int_0^{\sigma} \frac{\sqrt{a^2+c^2+2ab\cos\frac{s}{a}+b^2\cos^2\frac{s}{a}}}{a^2+b^2+c^2+2ab\cos\frac{s}{a}}ds \\
\Theta &= \int_0^{2\pi a} \frac{\sqrt{a^2+c^2+2ab\cos\frac{s}{a}+b^2\cos^2\frac{s}{a}}}{a^2+b^2+c^2+2ab\cos\frac{s}{a}}ds
\end{aligned}$$

There is a closed form for T but it is not helpful. Thus the following numerical approach is useful. Select sufficiently large N and for $j = 0, \ldots, N$, define

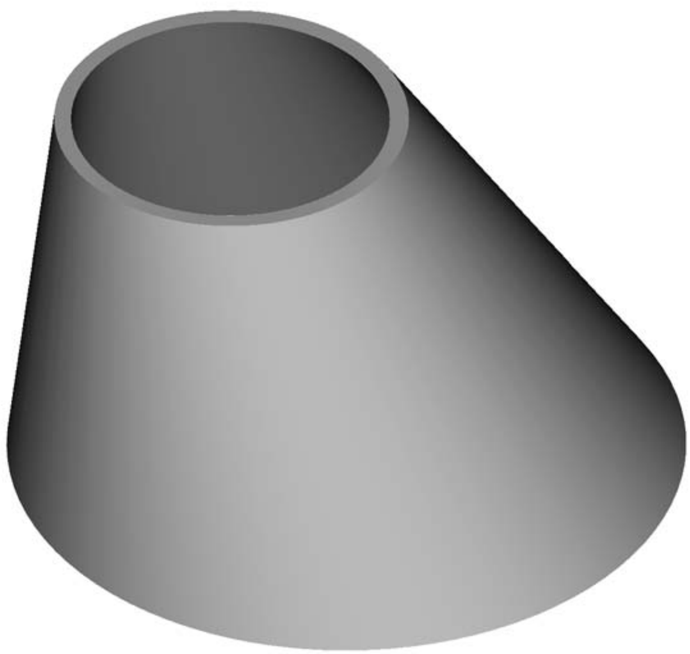
$$\begin{aligned}
\sigma_j &= \frac{jL}{N} \\
\theta_j &= \int_0^{\sigma_j} \frac{\sqrt{a^2+c^2+2ab\cos\frac{s}{a}+b^2\cos^2\frac{s}{a}}}{a^2+b^2+c^2+2ab\cos\frac{s}{a}}ds \\
\rho_j &= \ell(\sigma_j)
\end{aligned}$$

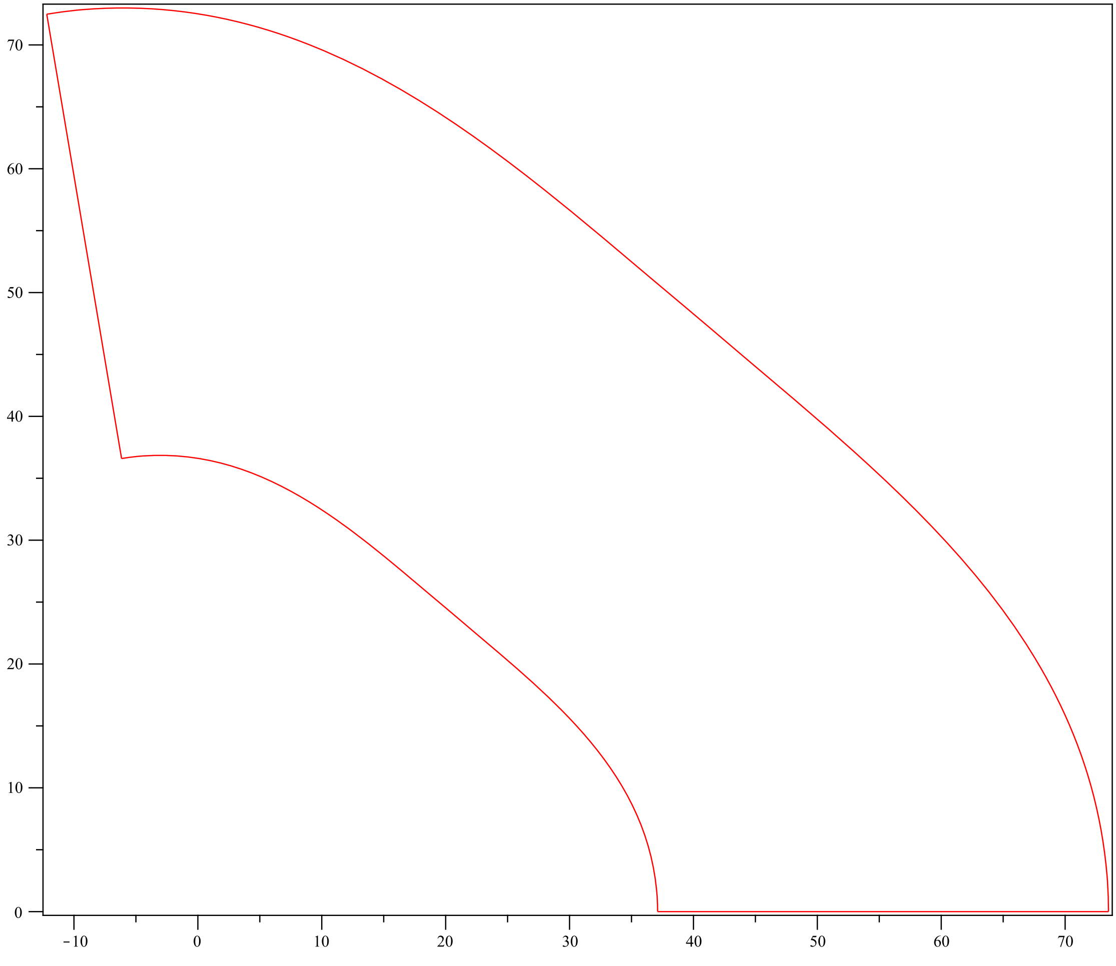
where of course the θ_j are computed numerically for some numerical selection of a, b, c. Then $\Theta = \theta_N$, and the sector is approximated by the pairs (ρ_j, θ_j).

5 Pictures and attachments

The three attachments to follow are:

1. CAD Picture of the cone.
2. The region described by the flattened out cone.
3. Maple script that computes equations for the flattened cone. Various 3D and 2D pictures are shown.





70
60
50
40
30
20
10
0
−10
0
10
20
30
40
50
60
70

Flattening a cone

Parametrizing a cone

set up cone and functions

> $Ps := \left\langle b + a\cdot\cos\left(\frac{s}{a}\right), a\cdot\sin\left(\frac{s}{a}\right), 0\right\rangle;$
> $Q := \langle 0, 0, c\rangle;$
> $C := (1-t)\cdot Q + t\cdot Ps;$
> $L := 2*\pi*a;$

$$Ps := \begin{bmatrix} b + a\cos\left(\frac{s}{a}\right) \\ a\sin\left(\frac{s}{a}\right) \\ 0 \end{bmatrix}$$

$$Q := \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

$$C := \begin{bmatrix} t\left(b + a\cos\left(\frac{s}{a}\right)\right) \\ t\,a\sin\left(\frac{s}{a}\right) \\ (1-t)\,c \end{bmatrix}$$

$$L := 2\,\pi\,a \tag{1.1.1}$$

```
> QPs := Ps-Q;
ls := sqrt(QPs[1]^2+QPs[2]^2+QPs[3]^2);
ls := simplify(Ls);
dls := simplify(diff(Ls, s));
```

$$QPs := \begin{bmatrix} b + a\cos\left(\frac{s}{a}\right) \\ a\sin\left(\frac{s}{a}\right) \\ -c \end{bmatrix}$$

$$ls := \sqrt{\left(b + a\cos\left(\frac{s}{a}\right)\right)^2 + a^2\sin\left(\frac{s}{a}\right)^2 + c^2}$$

$$ls := Ls$$
$$dls := 0 \quad \textbf{(1.1.2)}$$

specific values

> $bigD := 38.3750;$
> $littleD := 19.000;$
> $h0 := 31.650;$
> $a0 := \frac{bigD}{2};$
> $b0 := a0;$
> $c0 := \frac{h0 \cdot bigD}{(bigD - littleD)};$
> $t0 := \frac{h0}{c0};$
> $L0 := subs(a = \mathrm{a0}, \mathrm{b} = \mathrm{b0}, \mathrm{c} = \mathrm{c0}, \mathrm{L});$
> $w := 50;$

$$bigD := 38.3750$$
$$littleD := 19.000$$
$$h0 := 31.650$$
$$a0 := 19.18750000$$
$$b0 := 19.18750000$$
$$c0 := 62.68741935$$
$$t0 := 0.5048859935$$
$$L0 := 38.37500000\,\pi$$
$$w := 50 \quad \textbf{(1.2.1)}$$

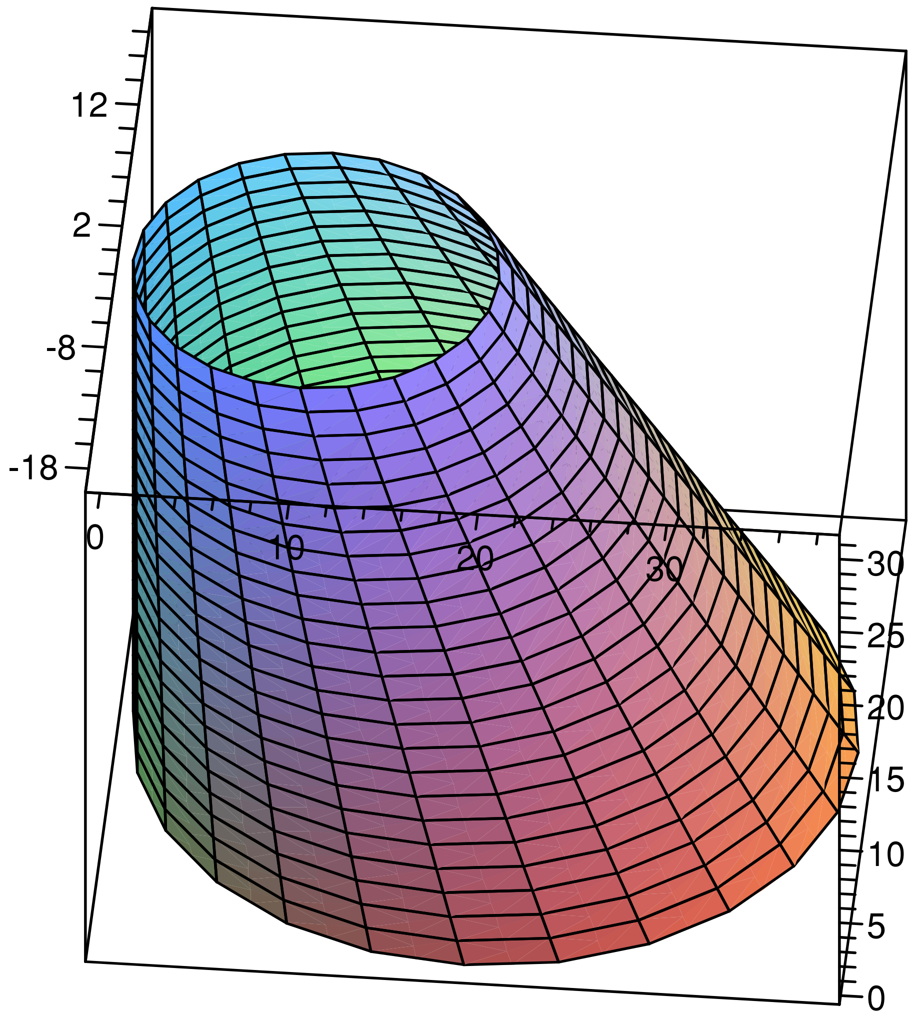
plot 3 views

> $\mathrm{C0} := subs(a = \mathrm{a0}, \mathrm{b} = \mathrm{b0}, \mathrm{c} = \mathrm{c0}, \mathrm{C});$
> $C0L := [C0[1], C0[2], C0[3]];$

$$C0 := \begin{bmatrix} t\,(19.18750000 + 19.18750000\cos(0.05211726384\,s)) \\ 19.18750000\,t\sin(0.05211726384\,s) \\ 62.68741935 - 62.68741935\,t \end{bmatrix}$$

$$C0L := [t\,(19.18750000 + 19.18750000\cos(0.05211726384\,s)), 19.18750000\,t\sin(0.05211726384\,s), 62.68741935 - 62.68741935\,t] \quad \textbf{(1.3.1)}$$

> $plot3d(\mathrm{C0L}, \mathrm{t} = \mathrm{t0}..1, \mathrm{s} = 0..\mathrm{L0}, orientation = [-85, 50], scaling = constrained, axes = boxed);$
> $plot3d(\mathrm{C0L}, \mathrm{t} = \mathrm{t0}..1, \mathrm{s} = 0..\mathrm{L0}, orientation = [-90, 0], scaling = constrained, axes = boxed);$
> $plot3d(\mathrm{C0L}, \mathrm{t} = \mathrm{t0}..1, \mathrm{s} = 0..\mathrm{L0}, orientation = [-85, 87], scaling = constrained, axes = boxed)\ ;$



12
2
-8
-18
0
10
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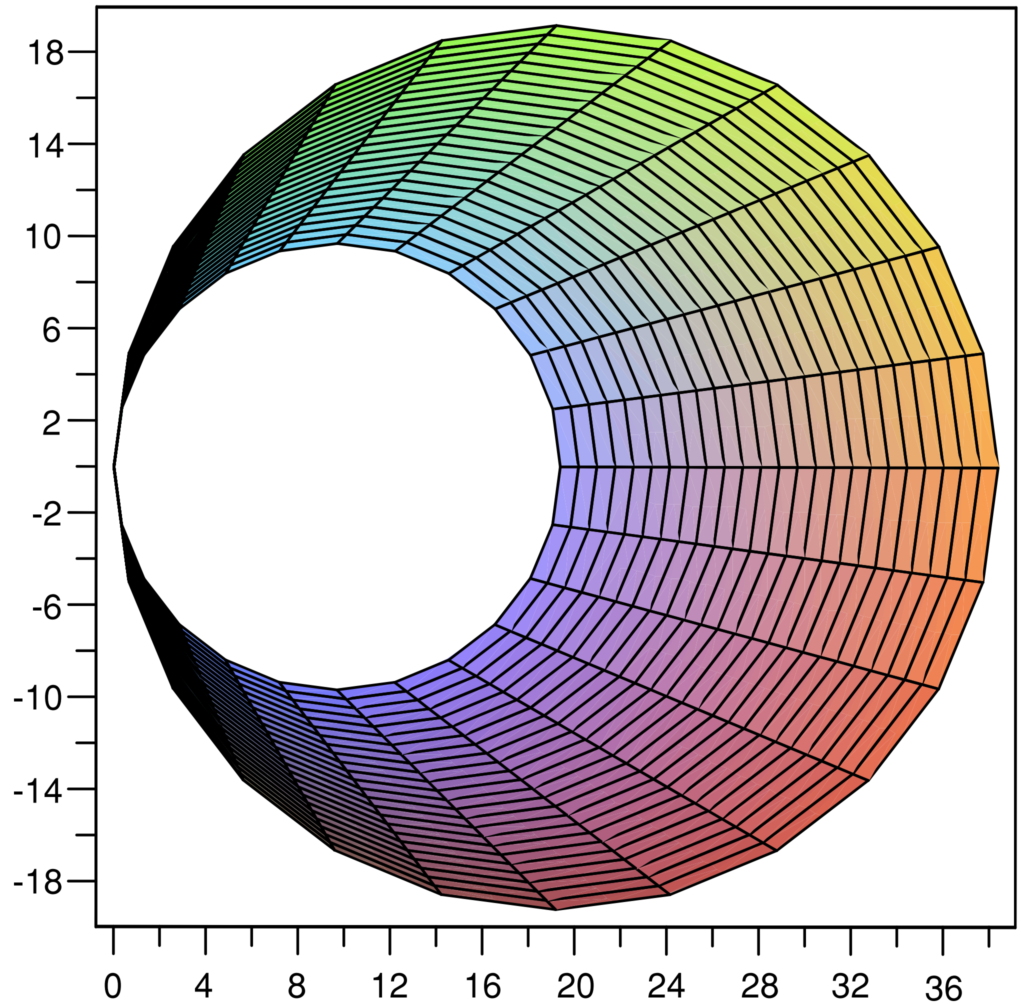
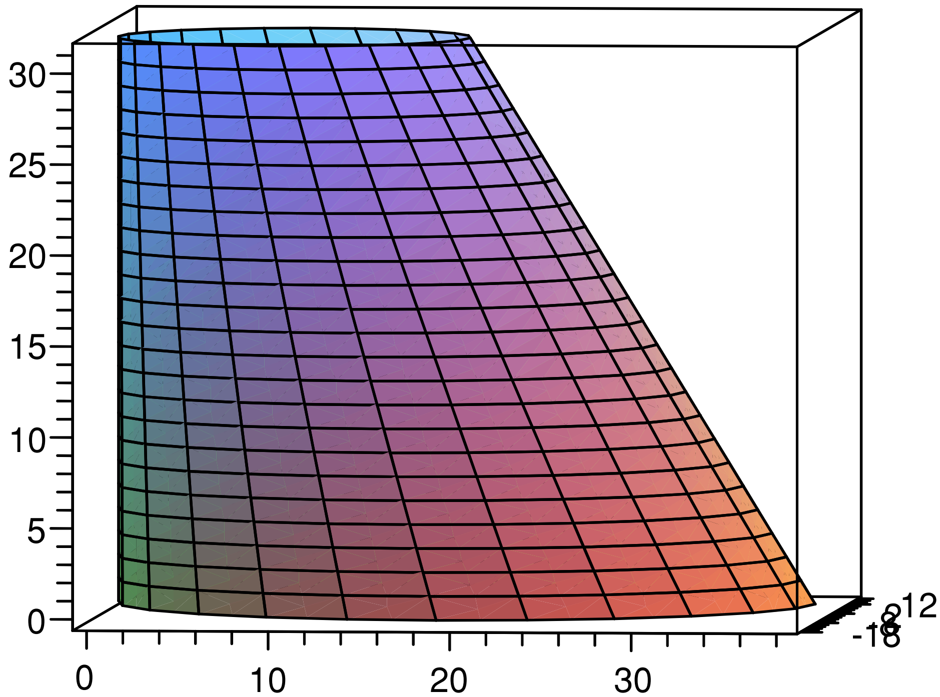
30
30
25
20
15
10
5
0


18
14
10
6
2
-2
-6
-10
-14
-18
0
4
8
12
16
20
24
28
32
36



Find the sector

find functions

```
> QPs := Ps - Q;
  ls := sqrt(QPs[1]^2 + QPs[2]^2 + QPs[3]^2);
  ls := simplify(ls);
  dls := simplify(diff(ls, s));
  Ks := sqrt(1 - dls^2) / ls;
  Ks := simplify(Ks);
  Ts := int(Ks, s);
```

$$QPs := \begin{bmatrix} b + a\cos\left(\frac{s}{a}\right) \\ a\sin\left(\frac{s}{a}\right) \\ -c \end{bmatrix}$$

$$ls := \sqrt{\left(b + a\cos\left(\frac{s}{a}\right)\right)^2 + a^2\sin\left(\frac{s}{a}\right)^2 + c^2}$$

$$ls := \sqrt{b^2 + 2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2}$$

$$dls := -\frac{b\sin\left(\frac{s}{a}\right)}{\sqrt{b^2 + 2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2}}$$

$$Ks := \frac{\sqrt{1 - \frac{b^2\sin\left(\frac{s}{a}\right)^2}{b^2 + 2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2}}}{\sqrt{b^2 + 2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2}}$$

$$Ks := \frac{\sqrt{\frac{2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2 + b^2\cos\left(\frac{s}{a}\right)^2}{b^2 + 2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2}}}{\sqrt{b^2 + 2\,b\,a\cos\left(\frac{s}{a}\right) + a^2 + c^2}}$$

$$Ts := \left(\sqrt{2}\left(\mathrm{EllipticF}\left(\frac{\sqrt{\frac{b^2 - c^2 - a^2 + 2\,\mathrm{I}\,b\,c}{b^2 + 2\,b\,a + a^2 + c^2}}\left(-1 + \cos\left(\frac{s}{a}\right)\right)}{\sin\left(\frac{s}{a}\right)},\right.\right.\right. \tag{2.1.1}$$

$$\left.\sqrt{\frac{2\,a^2c^2 - 2\,b^2a^2 - 6\,c^2b^2 + b^4 + c^4 + a^4 - 4\,\mathrm{I}\,b^3c + 4\,\mathrm{I}\,c^3b + 4\,\mathrm{I}\,a^2b\,c}{\left(b^2 - 2\,b\,a + a^2 + c^2\right)\left(b^2 + 2\,b\,a + a^2 + c^2\right)}}\right)a$$

$$-\,b\,\mathrm{EllipticPi}\left(\frac{\sqrt{\frac{b^2 - c^2 - a^2 + 2\,\mathrm{I}\,b\,c}{b^2 + 2\,b\,a + a^2 + c^2}}\left(-1 + \cos\left(\frac{s}{a}\right)\right)}{\sin\left(\frac{s}{a}\right)},\right.$$

$$-\frac{b^2-2ba+a^2+c^2}{b^2-c^2-a^2+2\,\mathrm{I}\,bc}, \frac{\sqrt{-\frac{-b^2+c^2+a^2+2\,\mathrm{I}\,bc}{b^2+2ba+a^2+c^2}}}{\sqrt{\frac{b^2-c^2-a^2+2\,\mathrm{I}\,bc}{b^2+2ba+a^2+c^2}}}\Bigg)$$

$$+ b\,\mathrm{EllipticPi}\left(\frac{\sqrt{\frac{b^2-c^2-a^2+2\,\mathrm{I}\,bc}{b^2+2ba+a^2+c^2}}\left(-1+\cos\left(\frac{s}{a}\right)\right)}{\sin\left(\frac{s}{a}\right)},\right.$$

$$-\frac{b^2+2ba+a^2+c^2}{b^2-c^2-a^2+2\,\mathrm{I}\,bc}, \frac{\sqrt{-\frac{-b^2+c^2+a^2+2\,\mathrm{I}\,bc}{b^2+2ba+a^2+c^2}}}{\sqrt{\frac{b^2-c^2-a^2+2\,\mathrm{I}\,bc}{b^2+2ba+a^2+c^2}}}\Bigg)\Bigg)$$

$$\sin\left(\frac{s}{a}\right)^2\sqrt{\frac{2ba\cos\left(\frac{s}{a}\right)+a^2+c^2+b^2\cos\left(\frac{s}{a}\right)^2}{b^2+2ba\cos\left(\frac{s}{a}\right)+a^2+c^2}}$$

$$\sqrt{-\frac{2\left(\mathrm{I}\,bc\cos\left(\frac{s}{a}\right)-b^2\cos\left(\frac{s}{a}\right)-ba\cos\left(\frac{s}{a}\right)-c^2-a^2-\mathrm{I}\,bc-ba\right)}{(b^2+2ba+a^2+c^2)\left(1+\cos\left(\frac{s}{a}\right)\right)}}$$

$$\sqrt{\frac{\mathrm{I}\,bc\cos\left(\frac{s}{a}\right)+b^2\cos\left(\frac{s}{a}\right)+ba\cos\left(\frac{s}{a}\right)+c^2+a^2-\mathrm{I}\,bc+ba}{(b^2+2ba+a^2+c^2)\left(1+\cos\left(\frac{s}{a}\right)\right)}}$$

$$\sqrt{b^2+2ba\cos\left(\frac{s}{a}\right)+a^2+c^2}\Bigg)\Bigg/$$

$$\left(\sqrt{\frac{b^2-c^2-a^2+2\,\mathrm{I}\,bc}{b^2+2ba+a^2+c^2}}\left(b^2\cos\left(\frac{s}{a}\right)^3-b^2\cos\left(\frac{s}{a}\right)^2+2ba\cos\left(\frac{s}{a}\right)^2\right.\right.$$

$$\left.\left.-2ba\cos\left(\frac{s}{a}\right)+a^2\cos\left(\frac{s}{a}\right)+c^2\cos\left(\frac{s}{a}\right)-a^2-c^2\right)\right)$$

```
> ls0 := subs(a = a0, b = b0, c = c0, ls);
  dls0 := subs(a = a0, b = b0, c = c0, dls);
```

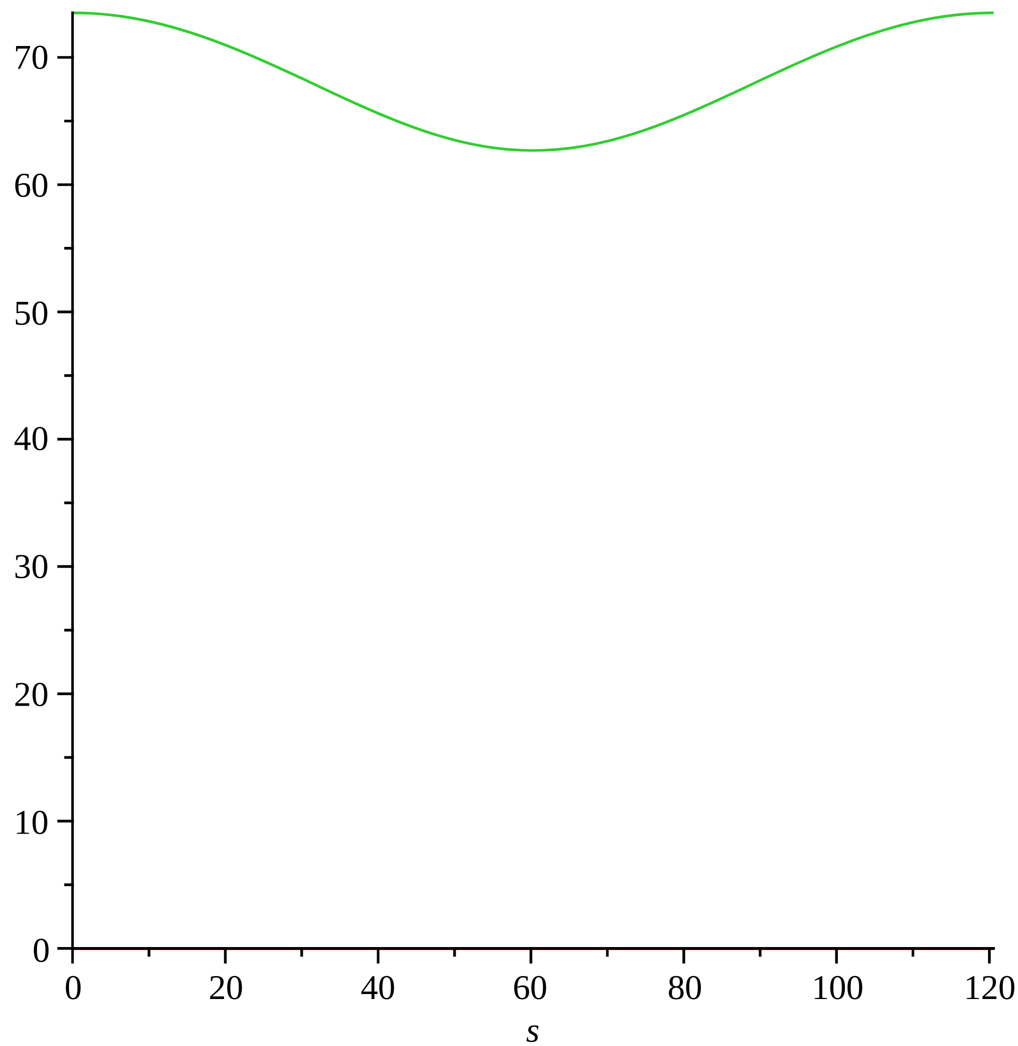
$$Ks0 := subs\left(a = a0,\ b = b0,\ c = c0,\ \frac{\mathrm{sqrt}(1 - dls^2)}{ls}\right);$$

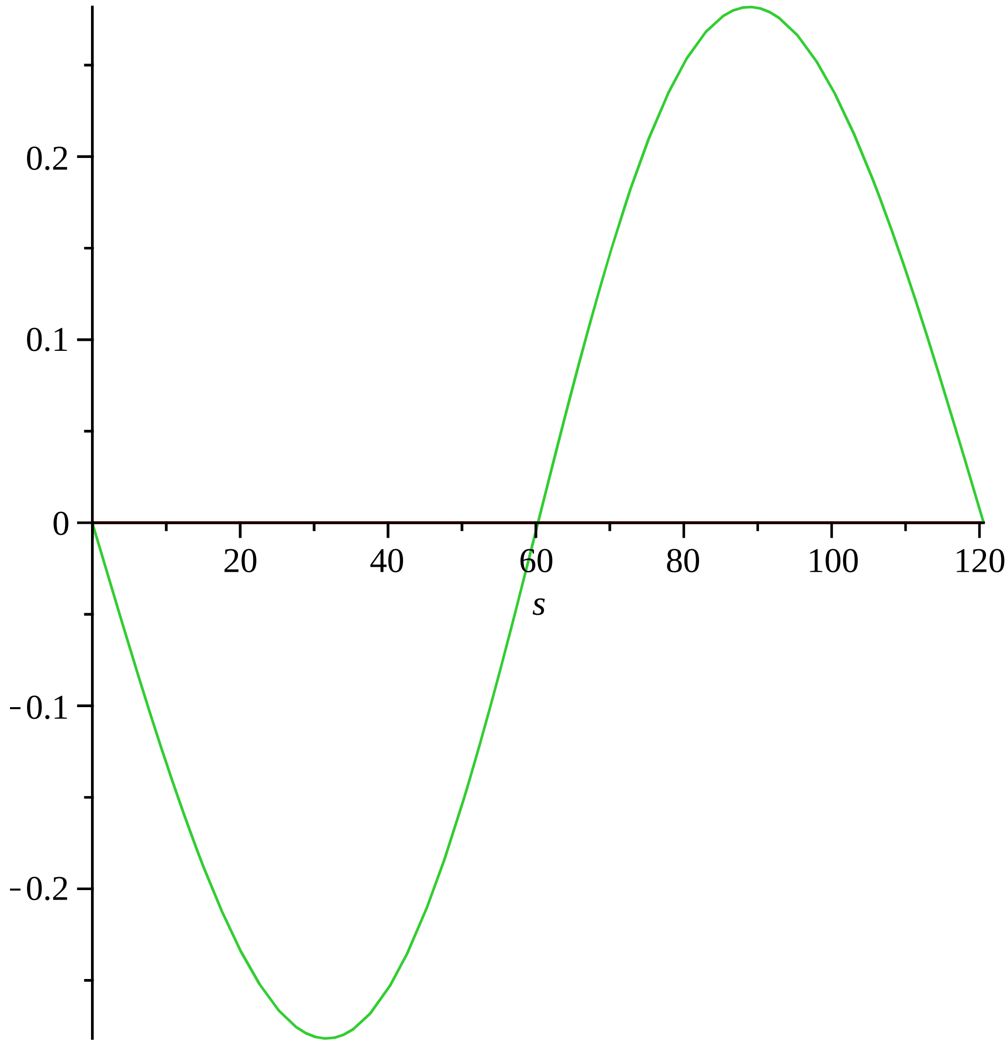
plot({0, ls0}, s = 0 ..L0);
plot({0, dls0}, s = 0 ..L0);
plot({0, Ks0}, s = 0 ..L0,);

$$ls0 := \sqrt{4666.032857 + 736.3203125 \cos(0.05211726384\, s)}$$

$$dls0 := -\frac{19.18750000 \sin(0.05211726384\, s)}{\sqrt{4666.032857 + 736.3203125 \cos(0.05211726384\, s)}}$$

$$Ks0 := \frac{\sqrt{1 - \frac{368.1601562 \sin(0.05211726384\, s)^2}{4666.032857 + 736.3203125 \cos(0.05211726384\, s)}}}{\sqrt{4666.032857 + 736.3203125 \cos(0.05211726384\, s)}}$$





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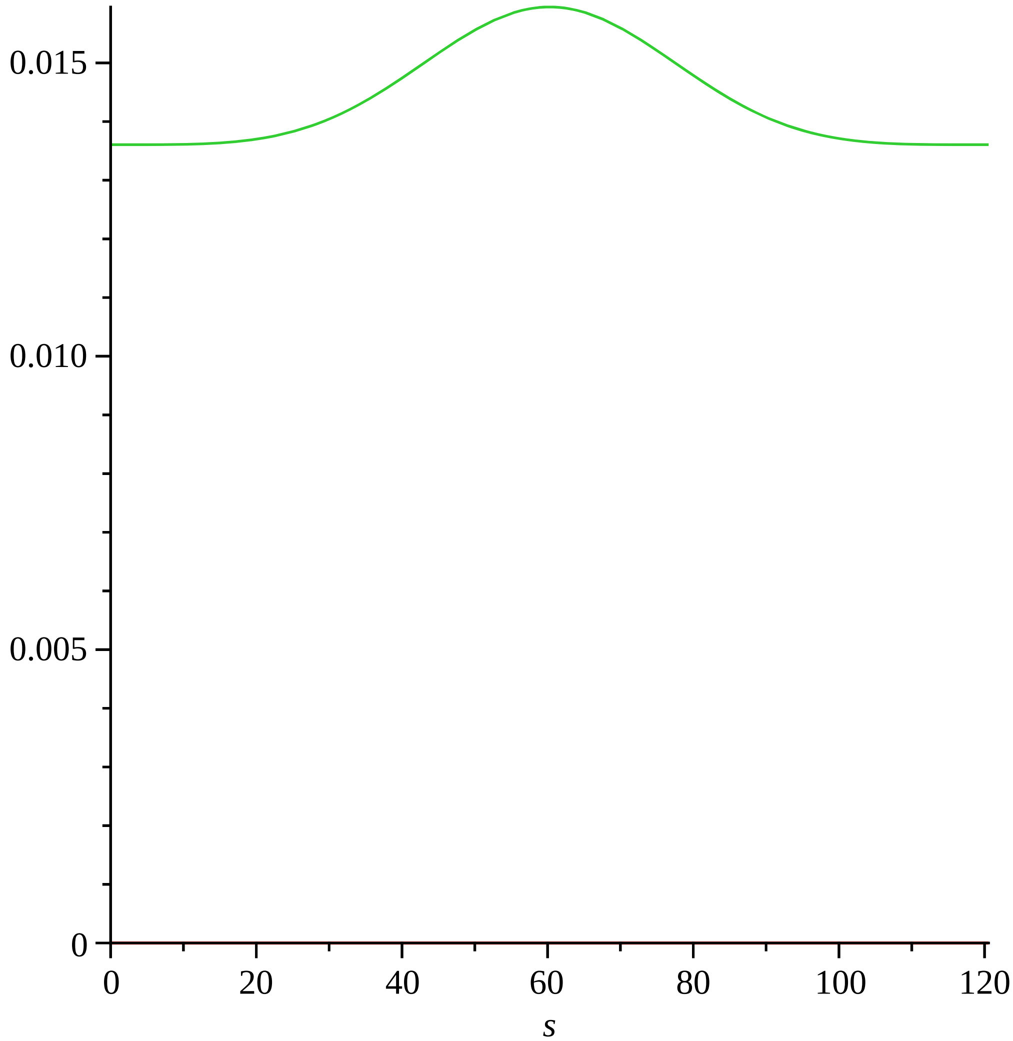
-0.1
-0.2
20
40
60
80
100
120
s



find and plot sector

> $\Theta := evalf(Int(Ks0, s = 0 ...L0));$
> $N := 100;$
> $ds := evalf\left(\frac{L0}{N}\right);$
> $spoints := [seq(j \cdot ds, j = 0 ..N)];$
> $Tsvals := [seq(evalf(Int(Ks0, s = 0 ..sj)), sj = spoints)];$
> $outrhovals := [seq(evalf(subs(s = sj, ls0)), sj = spoints)];$
> $inrhovals := [seq(t0 \cdot outrhovals[j], j = 1 ..N + 1)];$

$$\Theta := 1.737502361$$

$$N := 100$$

$$ds := 1.205586181$$

spoints := [0., 1.205586181, 2.411172362, 3.616758543, 4.822344724, 6.027930905, 7.233517086, 8.439103267, 9.644689448, 10.85027563, 12.05586181, 13.26144799, 14.46703417, 15.67262035, 16.87820653, 18.08379272, 19.28937890, 20.49496508, 21.70055126, 22.90613744, 24.11172362,

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115.7362734, 116.9418596, 118.1474457, 119.3530319, 120.5586181]

Tsvals := [0., 0.01640237677, 0.03280476304, 0.04920720628, 0.06560982977,
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outrhovals := [73.50070183, 73.49081718, 73.46119427, 73.41192622, 73.34316814,
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73.25513695, 73.34316814, 73.41192623, 73.46119427, 73.49081718, 73.50070183]

inrhovals := [37.10947487, 37.10448425, 37.08952805, 37.06465330, 37.02993831, **(2.2.1)**
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36.98549260, 37.02993831, 37.06465331, 37.08952805, 37.10448425, 37.10947487]

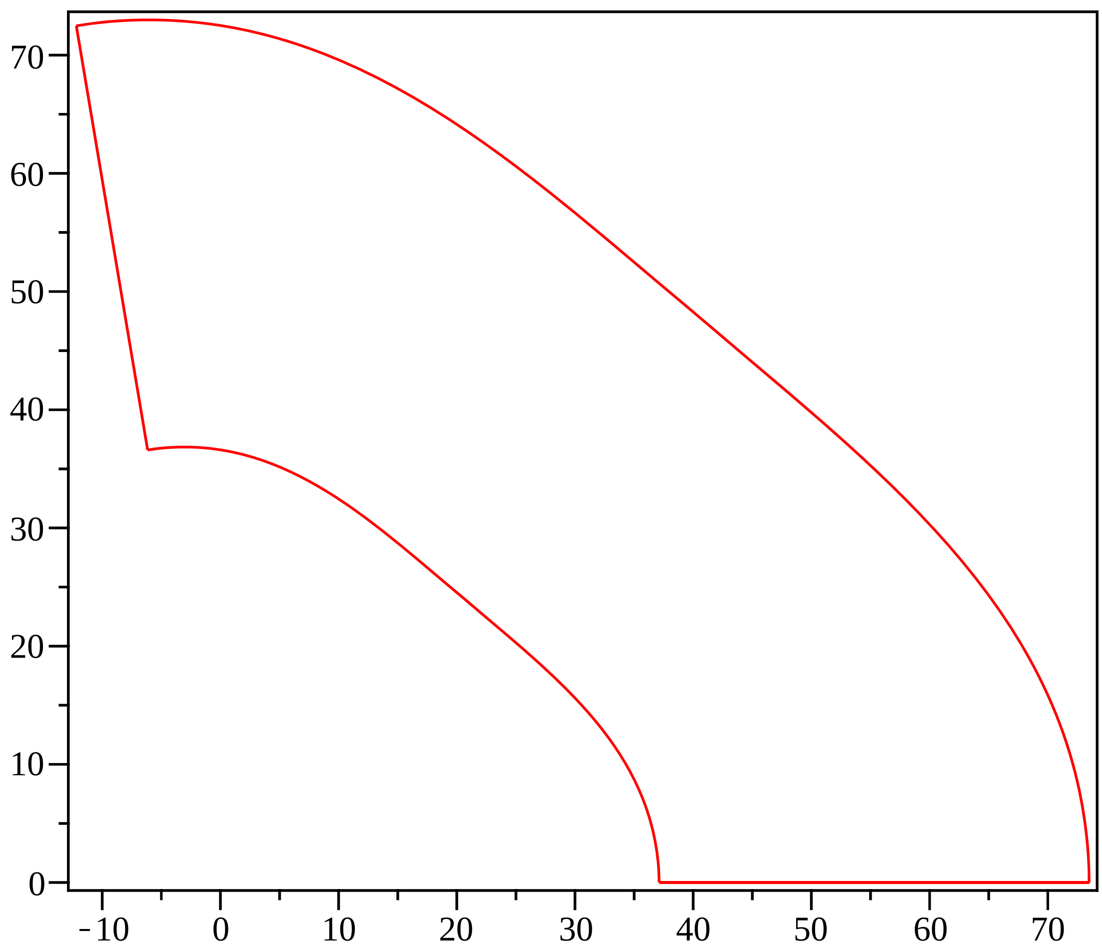
```
> with(plots) :
outpoints := [seq([outrhovals[j]·cos(Tsvals[j]), outrhovals[j]·sin(Tsvals[j])], j = 1
    ..N + 1)];
inpoints := [seq([inrhovals[j]·cos(Tsvals[j]), inrhovals[j]·sin(Tsvals[j])], j = 1 ..N
    + 1)];
line1 := [inpoints[1], outpoints[1]];
line2 := [inpoints[N + 1], outpoints[N + 1]];
display({plot(outpoints, scaling = constrained), plot(inpoints, scaling = constrained),
   plot(line1, scaling = constrained), plot(line2, scaling = constrained)}, axes = boxed,
   );
```

outpoints := [[73.50070183, 0.], [73.48093149, 1.205370022], [73.42167009, 2.409444862], [73.32306620, 3.610938161], [73.18536643, 4.808581167], [73.00891391, 6.001131369], [72.79414623, 7.187380990], [72.54159273, 8.366165216], [72.25187133, 9.536370142], [71.92568475, 10.69694036], [71.56381622, 11.84688615], [71.16712478, 12.98529009], [70.73654008, 14.11131329], [70.27305674, 15.22420084], [69.77772852, 16.32328672], [69.25166197, 17.40799791], [68.69600997, 18.47785768], [68.11196502, 19.53248816], [67.50075244, 20.57161193], [66.86362349, 21.59505259], [66.20184842, 22.60273462], [65.51670975, 23.59468195], [64.80949552, 24.57101576], [64.08149288, 25.53195121], [63.33398186, 26.47779322], [62.56822952, 27.40893126], [61.78548441, 28.32583340], [60.98697159, 29.22903945], [60.17388787, 30.11915350], [59.34739772, 30.99683575], [58.50862949, 31.86279402], [57.65867230, 32.71777470], [56.79857307, 33.56255373], [55.92933432, 34.39792732], [55.05191204, 35.22470277], [54.16721413, 36.04368946], [53.27609899, 36.85569022], [52.37937444, 37.66149295], [51.47779676, 38.46186280], [50.57206992, 39.25753494], [49.66284491, 40.04920781], [48.75071910, 40.83753719], [47.83623563, 41.62313070], [46.91988290, 42.40654322], [46.00209388, 43.18827274], [45.08324559, 43.96875702], [44.16365841, 44.74837066], [43.24359562, 45.52742294], [42.32326266, 46.30615605], [41.40280672, 47.08474381], [40.48231631, 47.86329083], [39.56182099, 48.64183203], [38.64129122, 49.42033249], [37.72063850, 50.19868756], [36.79971588, 50.97672324], [35.87831860, 51.75419675], [34.95618537, 52.53079715], [34.03299993, 53.30614632], [33.10839313, 54.07979986], [32.18194579, 54.85124813], [31.25319167, 55.61991749], [30.32162178, 56.38517145], [29.38668856, 57.14631209], [28.44781123, 57.90258159], [27.50438163, 58.65316374], [26.55577073, 59.39718578], [25.60133555, 60.13372034], [24.64042696, 60.86178768], [23.67239727, 61.58035826], [22.69660916, 62.28835529], [21.71244373, 62.98465809], [20.71930953, 63.66810547], [19.71665128, 64.33749958], [18.70395811, 64.99161024], [17.68077226, 65.62917960], [16.64669664, 66.24892714], [15.60140221, 66.84955528], [14.54463444, 67.42975510], [13.47621964, 67.98821243], [12.39606950, 68.52361442], [11.30418537, 69.03465608], [10.20066153, 69.52004709], [9.085687168, 69.97851875], [7.959547592, 70.40883093], [6.822624224, 70.80977905], [5.675394168, 71.18020076], [4.518427841, 71.51898279], [3.352386736, 71.82506733], [2.178019337, 72.09745832], [0.9961574395, 72.33522709], [−0.1922894465, 72.53751807], [−1.386339848, 72.70355371], [−2.584946480, 72.83263910], [−3.787003358, 72.92416609], [−4.991353204, 72.97761687], [−6.196795013, 72.99256689], [−7.402092150, 72.96868743], [−8.605981116, 72.90574739], [−9.807179495, 72.80361457], [−11.00439547, 72.66225629], [−12.19633569, 72.48173953]]

inpoints := [[37.10947487, 0.], [37.09949310, 0.6085744413], [37.06957285, 1.216494963], [37.01978912, 1.823112101], [36.95026644, 2.427785280], [36.86117803, 3.029887173], [36.75274484, 3.628807992], [36.62523412, 4.223959637], [36.47895784, 4.814779713], [36.31427081, 5.400735364], [36.13156845, 5.981326885], [35.93128450, 6.556091091], [35.71388831, 7.124604432], [35.47988207, 7.686485768], [35.22979779, 8.241398835], [34.96419416, 8.789054319], [34.68365324, 9.329211530], [34.38877713, 9.861679693], [34.08018446, 10.38631873], [33.75850697, 10.90303958], [33.42438601, 11.41180412], [33.07846909, 11.91262444], [32.72140653, 12.40556170], [32.35384820, 12.89072456], [31.97644035, 13.36826694], [31.58982272, 13.83838549], [31.19462568, 14.30131654], [30.79146775, 14.75733262], [30.38095316, 15.20673874], [29.96366986, 15.64986821], [29.54018753, 16.08707841], [29.11105604, 16.51874618], [28.67680400, 16.94526329], [28.23793752, 17.36703171], [27.79493930, 17.78445905], [27.34826772, 18.19795396], [26.89835617, 18.60792178], [26.44561250, 19.01476028], [25.99041856, 19.41885581], [25.53312976, 19.82057953], [25.07407479, 20.22028408], [24.61355524, 20.61830053], [24.15184535, 21.01493569], [23.68919170, 21.41046970], [23.22581287, 21.80515399], [22.76189924, 22.19920957], [22.29761256, 22.59282558], [21.83308574, 22.98615816], [21.36842252, 23.37932961], [20.90369720, 23.77242766], [20.43895449, 24.16550514], [19.97420930, 24.55857969], [19.50944671, 24.95163367], [19.04462205, 25.34461424], [18.57966111, 25.73743356], [18.11446053, 26.12996904], [17.64888838, 26.52206371], [17.18278498, 26.91352664], [16.71596396, 27.30413348], [16.24821367, 27.69362690], [15.77929873, 28.08171730], [15.30896214, 28.46808330], [14.83692745, 28.85237256], [14.36290144, 29.23420243], [13.88657704, 29.61316085], [13.40763669, 29.98880716], [12.92575573, 30.36067314], [12.44060645, 30.72826414], [11.95186182, 31.09106036], [11.45920006, 31.44851815], [10.96230873, 31.80007168], [10.46088918, 32.14513469], [9.954661069, 32.48310240], [9.443366474, 32.81335371], [8.926774265, 33.13525354], [8.404683971, 33.44815540], [7.876929455, 33.75140413], [7.343382209, 34.04433889], [6.803954539, 34.32629618], [6.258601865, 34.59661315], [5.707324864, 34.85463093], [5.150171132, 35.09969804], [4.587236192, 35.33117396], [4.018664093, 35.54843255], [3.444647410, 35.75086564], [2.865427023, 35.93788637], [2.281290930, 36.10893268], [1.692573108, 36.26347048], [1.099651457, 36.40099687], [0.5029459386, 36.52104300], [−0.09708424827, 36.62317688], [−0.6999435717, 36.70700595], [−1.305103272, 36.77217935], [−1.912004953, 36.81839004], [−2.520064321, 36.84537660], [−3.128675007, 36.85292466], [−3.737212649, 36.84086824], [−4.345039326, 36.80909070], [−4.951507562, 36.75752527], [−5.555965140, 36.68615546], [−6.157759065, 36.59501508]]

$line1 := [[37.10947487, 0.], [73.50070183, 0.]]$

$line2 := [[-6.157759065, 36.59501508], [-12.19633569, 72.48173953]]$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