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Bryan A. Curtis
Metropolitan State University of Denver

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Bryan A. Curtis[a]

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Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
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[a]Metropolitan State University of Denver

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Reversing A Doodle

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Abstract. The radius r neighborhood of a set X, denoted $N_r(X)$, is the collection of points within a distance r of X. We discuss some of the properties preserved by the radius r neighborhood in $\mathbb{R}^n$. In particular, we find a collection of sets which have a unique pre-image when mapped under N_r. This problem has interesting ties to convex geometry.

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1 The Doodle

A doodle is usually nothing more than a simple drawing scribbled in the margins of a school notebook. While doodling is probably not the best way to spend time in class, it can lead to some very interesting mathematics. In this paper, we investigate a particular type of doodle described by Ravi Vakil in *The Mathematics of Doodling* [4]. Starting with any image on a piece of paper, a *doodle* is formed by repeatedly drawing closed loops of uniform distance about the previous image on the page (see Figure 1). With each additional loop, a doodle appears more circular and encloses a larger area. In order to capture these qualities, Vakil defined the *radius r neighborhood.* While Vakil's definition was primarily intended for planar sets, he provided a natural generalization of the radius r neighborhood to n-dimensional sets. It is worth noting that the radius r neighborhood can also be modified for any space with a notion of distance. However, some of the results in this paper rely on various properties of $\mathbb{R}^n$.

Definition 1.1. Let $X \subseteq \mathbb{R}^n$ and $r \geq 0$. The radius r neighborhood of X is the set

$$N_r(X) = \{\boldsymbol{p} : \|\boldsymbol{p} - \boldsymbol{x}\| \leq r \text{ for some } \boldsymbol{x} \in X\}.$$

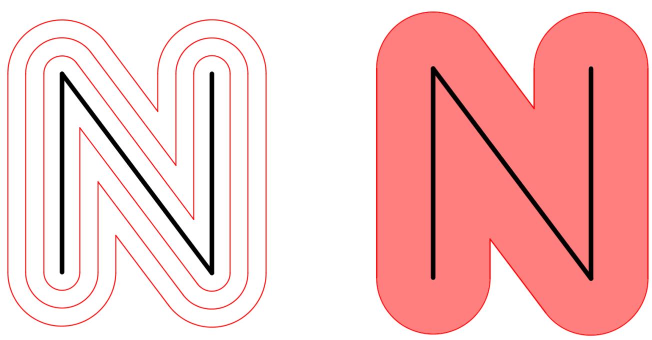

Figure 1: Left: A doodle repeated 3 times about the letter "N". Right: The corresponding radius r neighborhood of "N".

Notice, that when working in $\mathbb{R}^2$, the boundary of the radius r neighborhood perfectly models the outermost loop of the corresponding doodle, as can be seen in Figure 1. Definition 1.1 also allows us to leave the plane so that we may consider objects embedded within higher dimensions. This observation leads to the n-dimensional analogue of doodles, which we shall refer to as n-doodles. For an example of an n-doodle, imagine a single point in $\mathbb{R}^3$. The radius r neighborhood of this point produces a closed ball and the 3-doodle is formed by concentric spheres.

Having established the concept of a doodle, it is reasonable to ask if doodling is a reversible process. One approach begins by erasing all but the outermost loop of the doodle. We then draw a closed loop interior to the outermost loop; the new loop is a uniform distance (the same distance used in creating the doodle) away from the outermost loop. Finally, erase the outermost loop and repeat (see Figure 2). We might expect this *un-doodle* to result in the original image. However, as seen in Figure 2, un-doodling often results in an image other than the original.

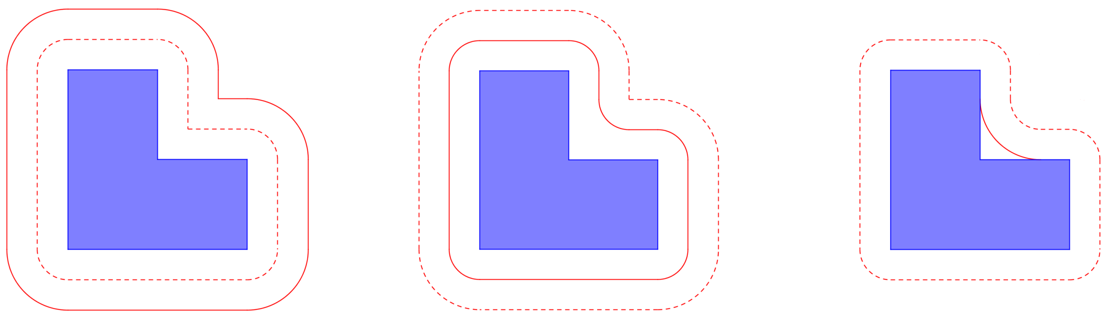

Figure 2: The above images depict the steps of an un-doodle in order from left to right. The solid line represents the current outermost loop and the dashed lines represent erased loops.

Definition 1.2. Let $X \subseteq \mathbb{R}^n$ and $r \geq 0$. The *radius r retraction* of X is the set

$$U_r(X) = \{\boldsymbol{p} : \|\boldsymbol{p} - \boldsymbol{u}\| > r \text{ for all points } \boldsymbol{u} \notin X\}.$$

The radius r retraction may be interpreted as a formalization of the un-doodling process described above. See Figure 3 for an example. We mention that for any set $X \in \mathbb{R}^n$ it can be shown $U_r(X) \subseteq X$. This result is not necessary for our purposes, but may be an insightful exercise for the reader (a similar statement is proved in Lemma 4.1).

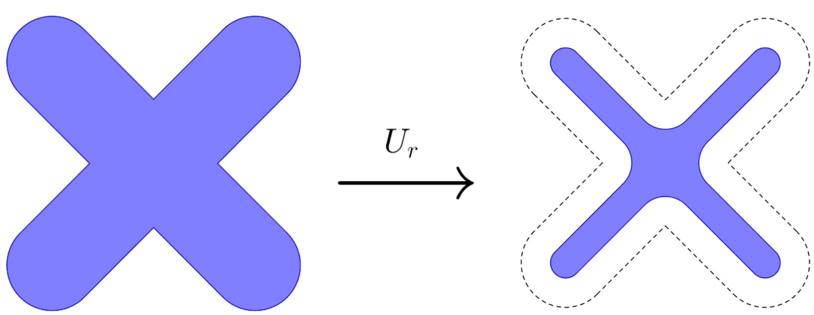

Figure 3: Left: The set "X". Right: The radius r retraction of "X". The dashed line is a reference to the boundary of the original set.

Of particular interest, Definition 1.2 will allow us to answer the question of when we can reverse an n-doodle (and consequently, when we can reverse a doodle). In terms of our formalization, we are looking for sets $X \in \mathbb{R}^n$ such that $N_r(X)$ has a unique pre-image. In this paper we show, that by restricting our attention to convex sets, $U_r(N_r(X)) = X$. See Theorem 4.2 for our formal statement.

The next section briefly discusses how our problem relates to convex sets. In Section 3, we momentarily drop the notion of a doodle in order to more rigorously examine the properties of the radius r neighborhood. Finally, in Section 4 we prove the main result of this paper.

2 Basic Properties and Convex Sets

In $\mathbb{R}^n$, we say that a set X is *convex*, if given points $\boldsymbol{x}, \boldsymbol{y} \in X$, the line segment $\overline{\boldsymbol{xy}}$ is also contained in X. Recall that the line segment $\overline{\boldsymbol{xy}}$ can be expressed as the set $\overline{\boldsymbol{xy}} = \{\boldsymbol{p} = (1-t)\boldsymbol{x} + t\boldsymbol{y} : 0 \leq t \leq 1\}$. Lemma 2.1 demonstrates how N_r preserves convexity (see Figure 4).

Lemma 2.1. *If a set X is convex then $N_r(X)$ is convex for all $r \geq 0$.*

Proof. Let $r \geq 0$. Assume that X is a convex set and let the points $\boldsymbol{p}_1, \boldsymbol{p}_2 \in N_r(X)$. Then there exist points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ such that $\|\boldsymbol{p}_1 - \boldsymbol{x}_1\| \leq r$ and $\|\boldsymbol{p}_2 - \boldsymbol{x}_2\| \leq r$. Further, since X is convex, we know that the line segment $\overline{\boldsymbol{x}_1\boldsymbol{x}_2} \subseteq X$.

In order to show $N_r(X)$ is convex, we must demonstrate that $\overline{\boldsymbol{p}_1\boldsymbol{p}_2} \subseteq N_r(X)$. To this end, let $\boldsymbol{u} \in \overline{\boldsymbol{p}_1\boldsymbol{p}_2}$. Then $\boldsymbol{u} = (1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2$ for some t satisfying $0 \leq t \leq 1$. Using this same value of t, define the point $\boldsymbol{v} = (1-t)\boldsymbol{x}_1 + t\boldsymbol{x}_2$. By this construction, $\boldsymbol{v} \in \overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ and consequently, $\boldsymbol{v} \in X$. Substituting $\boldsymbol{u}$ and $\boldsymbol{v}$ with their equivalent expressions and applying the triangle inequality, we obtain

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{v}\| &= \|(1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2 - (1-t)\boldsymbol{x}_1 - t\boldsymbol{x}_2\| \\
&\leq \|(1-t)(\boldsymbol{p}_1 - \boldsymbol{x}_1)\| + \|t(\boldsymbol{p}_2 - \boldsymbol{x}_2)\| \\
&\leq (1-t)r + tr = r.
\end{aligned}$$

Therefore, $\boldsymbol{u} \in N_r(X)$ and it follows that $\overline{\boldsymbol{p}_1\boldsymbol{p}_2} \subseteq N_r(X)$. We may conclude that $N_r(X)$ is convex. ☐

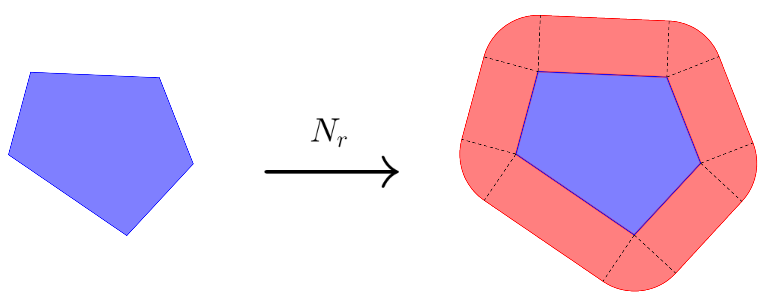

Figure 4: N_r maps a convex polygon to a convex shape.

In general, the converse of Lemma 2.1 is not true. Consider the annulus in Figure 5 with inner radius r. The radius r neighborhood of this annulus fills in the center and results in a convex disk. However, the annulus itself is not convex. In fact, if we were given the same annulus union with any subset of its inner circle, the radius r neighborhood would have

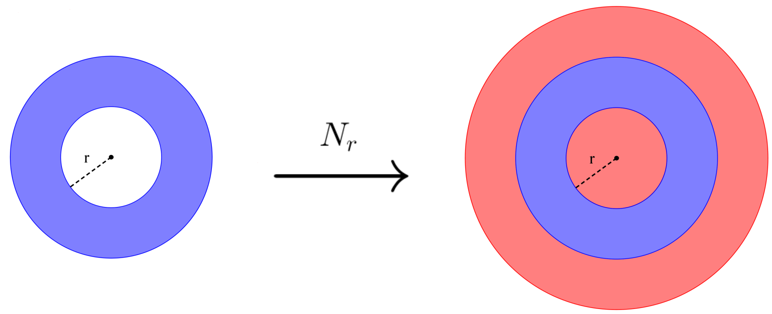

Figure 5: A pre-image of a convex radius r neighborhood is not necessarily convex.

resulted in the same convex disk. With this example, it becomes apparent that the radius r neighborhood about a set may not, in general, have a unique pre-image. Thus, in order to have a unique reversal of an n-doodle, we restrict ourselves to convex sets.

3 Geometric and Topological Considerations

We begin this section by reviewing some of the relevant terminology. For any $\boldsymbol{u} \in \mathbb{R}^n$ and $\epsilon > 0$, the *open ball* about $\boldsymbol{u}$ of radius ϵ, written $B_\epsilon(\boldsymbol{u})$, is given by

$$B_\epsilon(\boldsymbol{u}) = \{\boldsymbol{v} : \|\boldsymbol{u} - \boldsymbol{v}\| < \epsilon\}.$$

Let X be a subset of $\mathbb{R}^n$. A point $\boldsymbol{x}$ is a *boundary point* of X if for every $\epsilon > 0$ the open ball $B_\epsilon(\boldsymbol{x})$ contains at least one point from X and one point from the compliment of X. The *boundary* of X, denoted $\partial(X)$, is the set of all boundary points of X. The *closure* of X, denoted $\mathrm{Cl}(X)$, is the union $X \bigcup \partial(X)$. If $X = \mathrm{Cl}(X)$, X is said to be *closed*. If X is nonempty, then for each point $\boldsymbol{u} \in \mathbb{R}^n$, we define the *distance from $\boldsymbol{u}$ to X* by

$$d(\boldsymbol{u}, X) = \inf\{\|\boldsymbol{u} - \boldsymbol{x}\| : \boldsymbol{x} \in X\},$$

where $\inf S$ is the infimum of the set S.

Proposition 3.1. *Let $X \subset \mathbb{R}^n$ and the point $\boldsymbol{u}$ be arbitrary and fixed. If X is closed, then $d(\boldsymbol{u}, X) = \|\boldsymbol{u} - \boldsymbol{x}\|$ for some point $\boldsymbol{x} \in X$.*

The proof of Proposition 3.1, in a more general setting, may be found as an exercise in [3]. We note that the corresponding statement of Proposition 3.1 for arbitrary metric spaces would require that X be compact. Since we are working in $\mathbb{R}^n$, we may relax our hypothesis.

Proposition 3.2. *Let the point $\boldsymbol{u} \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. Then $d(\boldsymbol{x}, X) = 0$ if and only if $\boldsymbol{x} \in Cl(X)$.*

Proposition 3.2 may also be found as an exercise in [3]. This subtle, but important fact, will appear in the final proof of Theorem 4.2. We now prove our first key result.

Lemma 3.3. *Let X be a set and $r \geq 0$. Then $Cl(N_r(X)) = N_r(Cl(X))$.*

Proof. Our claim is true if $X = \emptyset$ since $\mathrm{Cl}(N_r(\emptyset)) = \emptyset = N_r(\mathrm{Cl}(\emptyset))$. Thus, we may now assume X is nonempty. We first demonstrate that $\mathrm{Cl}(N_r(X)) \subseteq N_r(\mathrm{Cl}(X))$. To this end, let $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$ and suppose, by way of contradiction, that $\boldsymbol{p} \notin N_r(\mathrm{Cl}(X))$. Then $d(\boldsymbol{p}, \mathrm{Cl}(X)) > r$ and consequently, we may find some $\epsilon > 0$ such that $d(\boldsymbol{p}, \mathrm{Cl}(X)) = r + \epsilon$.

Having fixed ϵ, we now show that $d(\boldsymbol{u}, \mathrm{Cl}(X)) > r$ for all $\boldsymbol{u} \in B_\epsilon(\boldsymbol{p})$. Again, we proceed by contradiction. Suppose there exists some $\boldsymbol{v} \in B_\epsilon(\boldsymbol{p})$ such that $d(\boldsymbol{v}, \mathrm{Cl}(X)) = t \leq r$. By Proposition 3.1, we may find a point $\boldsymbol{x} \in \mathrm{Cl}(X)$ such that $\|\boldsymbol{v} - \boldsymbol{x}\| = t$. Then

$$\|\boldsymbol{p} - \boldsymbol{x}\| = \|\boldsymbol{p} - \boldsymbol{v} + \boldsymbol{v} - \boldsymbol{x}\| \leq \|\boldsymbol{p} - \boldsymbol{v}\| + \|\boldsymbol{v} - \boldsymbol{x}\| < \epsilon + t \leq r + \epsilon$$

However, this is a contradiction, since we already established that $d(\boldsymbol{p}, \mathrm{Cl}(X)) = r + \epsilon$. Therefore, $d(\boldsymbol{u}, \mathrm{Cl}(X)) > r$ for all $\boldsymbol{u} \in B_\epsilon(\boldsymbol{p})$.

Since $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$, we may find some point $\boldsymbol{p'} \in B_\epsilon(\boldsymbol{p})$ such that $\boldsymbol{p'} \in N_r(X)$. However, this means that there exists some $\boldsymbol{x'} \in X$ such that $\|\boldsymbol{p'} - \boldsymbol{x'}\| \leq r$. Since $\boldsymbol{x'} \in \mathrm{Cl}(X)$, it follows that $d(\boldsymbol{p'}, \mathrm{Cl}(X)) \leq r$. However, this is a contradiction since $d(\boldsymbol{u}, \mathrm{Cl}(X)) > r$ for all $\boldsymbol{u} \in B_\epsilon(\boldsymbol{p})$. Therefore, it must be true that $\boldsymbol{p} \in N_r(\mathrm{Cl}(X))$ and consequently, $\mathrm{Cl}(N_r(X)) \subseteq N_r(\mathrm{Cl}(X))$.

We now show that $N_r(\mathrm{Cl}(X)) \subseteq \mathrm{Cl}(N_r(X))$. Let the point $\boldsymbol{p} \in N_r(\mathrm{Cl}(X))$. Then there exists a point $\boldsymbol{x} \in \mathrm{Cl}(X)$ such that $\|\boldsymbol{p} - \boldsymbol{x}\| \leq r$. If $\boldsymbol{x} \in X$, then $\boldsymbol{p} \in N_r(X)$ and since $N_r(X) \subseteq \mathrm{Cl}(N_r(X))$, it follows that $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$. Otherwise, $\boldsymbol{x} \in \partial(X)$. Let $\epsilon > 0$. Then there exists a point $\boldsymbol{x'} \in B_\epsilon(\boldsymbol{x})$ such that $\boldsymbol{x'} \in X$. It follows that

$$\|\boldsymbol{p} - \boldsymbol{x'}\| = \|\boldsymbol{p} - \boldsymbol{x} + \boldsymbol{x} - \boldsymbol{x'}\| \leq \|\boldsymbol{p} - \boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{x'}\| < r + \epsilon.$$

Define the point $\boldsymbol{p'} = \frac{r}{r+\epsilon}\boldsymbol{p} + \frac{\epsilon}{r+\epsilon}\boldsymbol{x'}$. Then

$$\|\boldsymbol{p} - \boldsymbol{p'}\| = \left\|\boldsymbol{p} - \frac{r}{r+\epsilon}\boldsymbol{p} - \frac{\epsilon}{r+\epsilon}\boldsymbol{x'}\right\| = \frac{\epsilon}{r+\epsilon}\|\boldsymbol{p} - \boldsymbol{x'}\| < \epsilon.$$

Therefore $\boldsymbol{p'} \in B_\epsilon(\boldsymbol{p})$ and

$$\|\boldsymbol{x'} - \boldsymbol{p'}\| = \|\boldsymbol{x'} - \frac{r}{r+\epsilon}\boldsymbol{p} - \frac{\epsilon}{r+\epsilon}\boldsymbol{x'}\| = \frac{r}{r+\epsilon}\|\boldsymbol{x'} - \boldsymbol{p}\| < r.$$

Thus $\boldsymbol{p'} \in N_r(X)$, and consequently $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$. It follows that $N_r(\mathrm{Cl}(X)) \subseteq \mathrm{Cl}(N_r(X))$ and we may conclude that $N_r(\mathrm{Cl}(X)) = \mathrm{Cl}(N_r(X))$. □

We now know that the radius r neighborhood and the closure of a set commute. This fact will allow us to demonstrate strong relationships between the boundary of a set X and the boundary of $N_r(X)$ in both Lemma 3.6 and Lemma 3.10. We must first prove the following lemma.

Lemma 3.4. *Let $r \geq 0$ and the point $\boldsymbol{p} \in \partial(N_r(X))$.*

(i) $\|\boldsymbol{p}-\boldsymbol{x}\| \geq r$ *for all* $\boldsymbol{x} \in Cl(X)$.

(ii) For all $\boldsymbol{x} \in Cl(X)$, *if* $\|\boldsymbol{p}-\boldsymbol{x}\| = r$ *then* $\boldsymbol{x} \in \partial(X)$.

Proof. Suppose there exists some point $\boldsymbol{y} \in \mathrm{Cl}(X)$ such that $\|\boldsymbol{p}-\boldsymbol{y}\| = s < r$. Let $\epsilon = r-s$. Since $\boldsymbol{y} \in \mathrm{Cl}(X)$, there exists a point $\boldsymbol{y}' \in B_{\epsilon/2}(\boldsymbol{y})$ also contained in X. Let the point $\boldsymbol{p}' \in B_{\epsilon/2}(\boldsymbol{p})$. Then

$$\begin{aligned}\|\boldsymbol{p}'-\boldsymbol{y}'\| &= \|\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{p}-\boldsymbol{y}+\boldsymbol{y}-\boldsymbol{y}'\| \\ &\leq \|\boldsymbol{p}'-\boldsymbol{p}\| + \|\boldsymbol{p}-\boldsymbol{y}\| + \|\boldsymbol{y}-\boldsymbol{y}'\| \\ &< \frac{r-s}{2} + s + \frac{r-s}{2} \\ &= r\end{aligned}$$

Therefore, $\boldsymbol{p}' \in N_r(X)$ and consequently, $B_{\epsilon/2}(\boldsymbol{p}) \subseteq N_r(X)$. This contradicts our assumption that $\boldsymbol{p} \in \partial(N_r(X))$ and it follows that $\|\boldsymbol{p}-\boldsymbol{x}\| \geq r$ for all $\boldsymbol{x} \in \mathrm{Cl}(X)$.

We now prove part (ii) of the lemma. Let $\boldsymbol{x} \in \mathrm{Cl}(X)$. Assume $\|\boldsymbol{p}-\boldsymbol{x}\| = r$ and suppose, by way of contradiction, that $\boldsymbol{x} \notin \partial(X)$. Then there exists a δ, satisfying $0 < \delta < r$, such that $B_\delta(\boldsymbol{x}) \subseteq X$. Let $s = \delta/2$ and define the point $\boldsymbol{x}' = \frac{r-s}{r}\boldsymbol{x} + \frac{s}{r}\boldsymbol{p}$. Then $\|\boldsymbol{x}-\boldsymbol{x}'\| = s$ so that $\boldsymbol{x}' \in X$. However, $\|\boldsymbol{p}-\boldsymbol{x}'\| = r-s < r$. This contradicts part (i) since $\boldsymbol{x}' \in X \subseteq \mathrm{Cl}(X)$. Thus, $\boldsymbol{x} \in \partial(X)$ and this completes the proof. □

The following proposition is a consequence of the Cauchy-Schwarz inequality and the triangle inequality.

Proposition 3.5. *Let the points* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be in* $\mathbb{R}^n$. *Then* $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ *if and only if there exists a* $t \geq 0$ *in* $\mathbb{R}$ *such that* $\boldsymbol{u} = t\boldsymbol{v}$.

Lemma 3.6. *Let* $X \subseteq \mathbb{R}^n$ *and* $r \geq 0$. *If* $\boldsymbol{p} \in \partial(N_r(X))$ *then there exists a point* $\boldsymbol{x} \in \partial(X)$ *such that* $\|\boldsymbol{p}-\boldsymbol{x}\| = r$. *Further, if* X *is convex then* $\boldsymbol{x}$ *is unique.*

Proof. Let the point $\boldsymbol{p} \in \partial(N_r(X))$. We first note that if $r = 0$ then $N_r(X) = X$. The claim follows with the observation that $\|\boldsymbol{p}-\boldsymbol{p}\| = 0$. Assume $r > 0$ for the remainder of the proof. Since $\boldsymbol{p} \in \partial(N_r(X))$, we know $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$ and by Lemma 3.3, it follows that $\boldsymbol{p} \in N_r(\mathrm{Cl}(X))$. Then there exists a point $\boldsymbol{x} \in \mathrm{Cl}(X)$ such that $\|\boldsymbol{p}-\boldsymbol{x}\| \leq r$. Further, by Lemma 3.4 (i), we may write this inequality as an equality, i.e. $\|\boldsymbol{p}-\boldsymbol{x}\| = r$. It remains to show that $\boldsymbol{x}$ is a boundary point of X.

Let $\delta > 0$ and $t = \min\{\frac{1}{2}, \frac{\delta}{2r}\}$. Define the point $\boldsymbol{u} = (1-t)\boldsymbol{x} + t\boldsymbol{p}$. Then $\boldsymbol{u} \in \overline{\boldsymbol{x}\boldsymbol{p}}$ and by the definition of t, we see that $0 < \|\boldsymbol{p}-\boldsymbol{u}\| < r$. Since $\boldsymbol{p}$ is a boundary point of $N_r(X)$, we know by Lemma 3.4 (i) that $\boldsymbol{u} \notin \mathrm{Cl}(X)$ and so also $u \notin X$. If $t = \frac{1}{2}$ then $\delta \geq r$ and

$$\|\boldsymbol{x}-\boldsymbol{u}\| = \left\|\frac{1}{2}\boldsymbol{x} - \frac{1}{2}\boldsymbol{p}\right\| = \frac{1}{2}r \leq \frac{\delta}{2}.$$

Otherwise, $t = \frac{\delta}{2r} < \frac{1}{2}$ and

$$\begin{aligned}\|\boldsymbol{x} - \boldsymbol{u}\| &= \left\|\boldsymbol{x} - \left(1 - \frac{\delta}{2r}\right)\boldsymbol{x} - \frac{\delta}{2r}\boldsymbol{p}\right\| \\ &= \left\|\frac{\delta}{2r}\boldsymbol{x} - \frac{\delta}{2r}\boldsymbol{p}\right\| \\ &= \frac{\delta}{2}.\end{aligned}$$

In both cases we have $\boldsymbol{u} \in B_\delta(\boldsymbol{x})$. This combined with the fact that $\boldsymbol{x} \in \mathrm{Cl}(X)$ guarantees that $\boldsymbol{x}$ is a boundary point of X.

We will now show that if X is convex then $\boldsymbol{x}$ is unique. Let the set X be convex. Assume $\boldsymbol{p} \in \partial N_r(X)$ and that there are two distinct points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \partial(X)$ such that $\|\boldsymbol{p} - \boldsymbol{x}_1\| = \|\boldsymbol{p} - \boldsymbol{x}_2\| = r$. By Proposition 3.5, the equality

$$\|\boldsymbol{p} - \boldsymbol{x}_1 + \boldsymbol{p} - \boldsymbol{x}_2\| = \|\boldsymbol{p} - \boldsymbol{x}_1\| + \|\boldsymbol{p} - \boldsymbol{x}_2\| \tag{1}$$

only holds if there exists a $t' \geq 0$ such that

$$\boldsymbol{p} - \boldsymbol{x}_1 = t'(\boldsymbol{p} - \boldsymbol{x}_2). \tag{2}$$

Having assumed $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are distinct, it follows that $t' \neq 1$. Thus, we may solve for $\boldsymbol{p}$ in equation (2), to obtain

$$\boldsymbol{p} = \frac{1}{1-t'}\boldsymbol{x}_1 - \frac{t'}{1-t'}\boldsymbol{x}_2.$$

After substituting this new expression for $\boldsymbol{p}$ into $\|\boldsymbol{p} - \boldsymbol{x}_1\| = \|\boldsymbol{p} - \boldsymbol{x}_2\|$ and after a bit of manipulation, we find that $|\frac{t'}{1-t'}| = |\frac{1}{1-t'}|$ which is impossible. It follows that the equality in (1) does not hold. In particular, $\|\boldsymbol{p} - \boldsymbol{x}_1 + \boldsymbol{p} - \boldsymbol{x}_2\| < \|\boldsymbol{p} - \boldsymbol{x}_1\| + \|\boldsymbol{p} - \boldsymbol{x}_2\|$. Define the point $\boldsymbol{v} = \frac{1}{2}\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2$. Then

$$\begin{aligned}\|\boldsymbol{p} - \boldsymbol{v}\| &= \left\|\boldsymbol{p} - \frac{1}{2}\boldsymbol{x}_1 - \frac{1}{2}\boldsymbol{x}_2\right\| \\ &= \left\|\frac{1}{2}\boldsymbol{p} - \frac{1}{2}\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{p} - \frac{1}{2}\boldsymbol{x}_2\right\| \\ &< \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{x}_1\| + \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{x}_2\| \\ &= r.\end{aligned}$$

Since X is convex, $\mathrm{Cl}(X)$ is convex and thus, the line segment $\overline{\boldsymbol{x}_1\boldsymbol{x}_2} \subseteq \mathrm{Cl}(X)$. However, $\boldsymbol{v} \in \overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ and consequently, $\boldsymbol{v} \in \mathrm{Cl}(X)$. This contradicts Lemma 3.4 (i) and we may conclude that if X is convex, then there exists a unique point $\boldsymbol{x} \in \partial(X)$ such that $\|\boldsymbol{p} - \boldsymbol{x}\| = r$. $\square$

While it may not be entirely obvious at first glance, the proof of Lemma 3.10 below requires that we take into consideration the dimension of $\mathbb{R}^n$. For our purposes, it will be enough to understand the geometry of $\mathbb{R}^n$. The following definitions and propositions may be found in [1] and [2]. The reader is encouraged to see these references for a more detailed discussion.

A translate of a set $X \subset \mathbb{R}^n$ by the point $\boldsymbol{y}$ is the set $X + \boldsymbol{y} = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in X\}$. A *hyperplane* is a translate of a subspace of $\mathbb{R}^n$ with codimension 1. That is, for some fixed point $\boldsymbol{y} \in \mathbb{R}^n$, a hyperplane H is the set $H = S + \boldsymbol{y}$, where S is a subspace of $\mathbb{R}^n$ with dimension $n-1$. For example, in $\mathbb{R}^2$, a hyperplane is a straight line.

One of the most important properties of hyperplanes is that they split $\mathbb{R}^n$ into two disjoint sets called *open half spaces*. A *closed half space* of a hyperplane H is the union of H and one of the corresponding half spaces.

Definition 3.7. Let $X \subset \mathbb{R}^n$. Then H is called a *supporting* hyperplane of X if

(i) X is contained in a closed half space of H and

(ii) there exists a point $\boldsymbol{x} \in \mathrm{Cl}(X)$ which is also contained in H.

For $\boldsymbol{x} \in \mathrm{Cl}(X)$, we say that H is a supporting hyperplane of X *at* $\boldsymbol{x}$ if $\boldsymbol{x}$ is contained in H and if X is contained in one of the two closed half spaces determined by H.

Proposition 3.8. *Let the set $X \subseteq \mathbb{R}^n$ be convex and let the point $\boldsymbol{x} \in \partial(X)$. Then there exists a supporting hyperplane of X at $\boldsymbol{x}$.*

A hyperplane H is said to be *parallel* to a hyperplane G if H is a translate of G. Without too much difficulty, it can be shown that for any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in H$, we have $d(\boldsymbol{x}_1, G) = d(\boldsymbol{x}_2, G)$. Thus, we may define the distance from H to G as $d(H, G) = d(\boldsymbol{x}, G)$ for any $\boldsymbol{x} \in H$ without any ambiguity. The following proposition is not provided in the references cited above. However, Proposition 3.9 can be derived from these sources. We omit the proof to avoid a lengthy digression.

Proposition 3.9. *Fix $r > 0$ and let H be a hyperplane. Then there exist exactly two distinct hyperplanes G_1 and G_2, each in a separate halfspace of H, such that the distance $d(H, G_1) = d(H, G_2) = r$.*

The reader is encouraged to check the validity of Proposition 3.9 in both $\mathbb{R}^2$ and $\mathbb{R}^3$ for an intuitive grasp of the statement. We are now ready to prove the final result of this section.

Lemma 3.10. *Let X be a convex set and $r \geq 0$. If the point $\boldsymbol{x} \in \partial(X)$, then there exists a point $\boldsymbol{p} \in \partial(N_r(X))$ such that $\|\boldsymbol{x} - \boldsymbol{p}\| = r$.*

Proof. The proof is immediate if $r = 0$. So, assume $r > 0$. Begin by letting the point $\boldsymbol{x} \in \partial(X)$. By Proposition 3.8, there exists a supporting hyperplane H of X at $\boldsymbol{x}$. We shall denote the two closed halfspaces of H by H_X and H_G, where H_X is the halfspace containing X. Proposition 3.9 guarantees the existence of a hyperplane G parallel to H, satisfying $d(H, G) = r$ and which is contained in H_G. By Proposition 3.1, there exists a point $\boldsymbol{p} \in G$

such that $\|\boldsymbol{x}-\boldsymbol{p}\| = r$. Since $\boldsymbol{x} \in \partial(X)$, we know $\boldsymbol{x} \in \mathrm{Cl}(X)$. Thus, $\boldsymbol{p} \in N_r(\mathrm{Cl}(X))$ and by Lemma 3.3, $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$. In fact, it will be shown that $\boldsymbol{p} \in \partial(N_r(X))$.

First, we demonstrate that G is a supporting hyperplane of $N_r(X)$. Since we already know $\boldsymbol{p} \in \mathrm{Cl}(N_r(X))$ and $\boldsymbol{p} \in G$, it remains to show $N_r(X)$ is contained in one of the closed half spaces of G (see Definition 3.7). By our construction of G, the closed halfspace H_X (which contains X) is contained in one of the closed halfspaces of G; call this halfspace G_X. Since $X \subseteq H_X$, it follows that for any $\boldsymbol{y} \in X$, we have $d(\boldsymbol{y}, G) \geq d(\boldsymbol{y}, H) + d(H, G) \geq r$. Consequently, $N_r(X) \subseteq G_X$ and we see that G is indeed a supporting hyperplane of $N_r(X)$.

To finish the proof, we must show $\boldsymbol{p} \in \partial(N_r(X))$. Suppose, this is not true. Then there exists an $\epsilon > 0$ such that $B_\epsilon(\boldsymbol{p}) \subseteq N_r(X)$. However, since $\boldsymbol{p} \in G$, $B_\epsilon(\boldsymbol{p})$ intersects nontrivially with both open halfspaces of G. This contradicts that G is a supporting hyperplane of $N_r(X)$, completing the proof. $\square$

Upon the first inspection of Lemma 3.10, it may not seem necessary that the set X be convex. However, it is this property that guarantees the existence of a supporting hyperplane at any boundary point. In fact, this statement can be generalized to any set $X \subset \mathbb{R}^n$ provided there exists a supporting hyperplane at the point $\boldsymbol{x} \in \partial(X)$. The annulus in Figure 5 demonstrates this quite nicely.

4 Reversing a Doodle

We are nearly ready to prove the main result of this paper. The reader should refer to Definition 1.2 for a reminder as to the meaning of U_r. As we will see, under the appropriate restrictions, U_r acts as the inverse of N_r.

Lemma 4.1. *Let $r \geq 0$ and $X \subseteq \mathbb{R}^n$. If the point $\boldsymbol{p} \in U_r(N_r(X))$ then $N_r(\{\boldsymbol{p}\}) \subseteq N_r(X)$.*

Proof. Begin by letting the point $\boldsymbol{p} \in U_r(N_r(X))$. Then $\|\boldsymbol{p}-\boldsymbol{u}\| > r$ for every point $\boldsymbol{u} \notin N_r(X)$. Now let the point $\boldsymbol{p}' \in N_r(\{\boldsymbol{p}\})$. Then $\|\boldsymbol{p}-\boldsymbol{p}'\| \leq r$ and consequently, $\boldsymbol{p}' \in N_r(X)$. Therefore, $N_r(\{\boldsymbol{p}\}) \subseteq N_r(X)$. $\square$

Theorem 4.2. *Let $r \geq 0$. If Cl(X) is a convex subset of $\mathbb{R}^n$ then $U_r(N_r(X)) = X$.*

Proof. Assume $\mathrm{Cl}(X)$ is a convex subset of $\mathbb{R}^n$. The proof is immediate if $X = \emptyset$, so we will assume that X is non-empty. Begin by letting $\boldsymbol{x} \in X$. If $N_r(X) = \mathbb{R}^n$ then there do not exist any points which are not contained in $N_r(X)$ and it follows that $U_r(N_r(X)) = \mathbb{R}^n$. Otherwise, let the point $\boldsymbol{u} \notin N_r(X)$. Then $\|\boldsymbol{x}-\boldsymbol{u}\| > r$. In both cases $\boldsymbol{x} \in U_r(N_r(X))$ and we may conclude that $X \subseteq U_r(N_r(X))$.

It remains to show $U_r(N_r(X)) \subseteq X$. Suppose that the point $\boldsymbol{p} \in U_r(N_r(X))$. By Proposition 3.1 we may find a point $\boldsymbol{x}_1 \in \mathrm{Cl}(X)$ such that

$$d(\boldsymbol{p}, \mathrm{Cl}(X)) = \|\boldsymbol{p}-\boldsymbol{x}_1\| = c.$$

We first demonstrate that $c = 0$, which will imply, by Proposition 3.2, that $\boldsymbol{p} \in \mathrm{Cl}(X)$. To this end, assume $c > 0$. Define the point $\boldsymbol{q} = \frac{r+c}{c}\boldsymbol{p} - \frac{r}{c}\boldsymbol{x}_1$. Then $\|\boldsymbol{p}-\boldsymbol{q}\| = r$ and consequently,

$\boldsymbol{q} \in N_r(\{\boldsymbol{p}\})$. By Lemma 4.1, we know $\boldsymbol{q} \in N_r(X)$. Then, there exists a point $\boldsymbol{x}_2 \in X$ such that $\|\boldsymbol{x}_2 - \boldsymbol{q}\| \leq r$. Define the point $\boldsymbol{y} = \frac{r}{r+c}\boldsymbol{x}_1 + \frac{c}{r+c}\boldsymbol{x}_2$. Then $\boldsymbol{y} \in \overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ and since $\mathrm{Cl}(X)$ is convex, $\boldsymbol{y} \in \mathrm{Cl}(X)$. Solving for $\boldsymbol{p}$ in the definition of $\boldsymbol{q}$ yields $\boldsymbol{p} = \frac{r}{r+c}\boldsymbol{x}_1 + \frac{c}{r+c}\boldsymbol{q}$. It follows that

$$\begin{aligned}
\|\boldsymbol{p} - \boldsymbol{y}\| &= \left\| \frac{r}{r+c}\boldsymbol{x}_1 + \frac{c}{r+c}\boldsymbol{q} - \left(\frac{r}{r+c}\boldsymbol{x}_1 + \frac{c}{r+c}\boldsymbol{x}_2 \right) \right\| \\
&= \frac{c}{r+c} \|\boldsymbol{q} - \boldsymbol{x}_2\| \\
&< c
\end{aligned}$$

However, this directly contradicts $d(\boldsymbol{p}, \mathrm{Cl}(X)) = c$. It follows that $c = 0$ and thus, $\boldsymbol{p} \in \mathrm{Cl}(X)$.

If $\boldsymbol{p} \in X$, we are done. Otherwise $\boldsymbol{p} \in \partial(X) \setminus X$. Then, by Lemma 3.10, there exists a point $\boldsymbol{p}' \in \partial(N_r(X))$ such that $\|\boldsymbol{p} - \boldsymbol{p}'\| = r$. Thus, by Lemma 4.1, we know $\boldsymbol{p}' \in N_r(X)$. This means that there exists some point $\boldsymbol{z} \in X$, such that $\|\boldsymbol{p}' - \boldsymbol{z}\| \leq r$. By Lemma 3.4 (i), we know $\|\boldsymbol{p}' - \boldsymbol{z}\| \geq r$ and by Lemma 3.4 (ii) if $\|\boldsymbol{p} - \boldsymbol{z}\| = r$ then $\boldsymbol{z} \in \partial(X)$. However, Lemma 3.6 guarantees that $\boldsymbol{p}$ is the only point in $\partial(X)$ that is a distance r from $\boldsymbol{p}'$. We have reached a contradiction since this implies that $\|\boldsymbol{p}' - \boldsymbol{z}\| > r$. It follows that $\boldsymbol{p} \in X$ and consequently, $U_r(N_r(X)) \subseteq X$. We may conclude that $U_r(N_r(X)) = X$. □

Since the closure of a convex set is also convex, we have the following corollary.

Corollary 4.3. *Let $r \geq 0$. If X is a convex subset of $\mathbb{R}^n$ then $U_r(N_r(X)) = X$.*

Remark 4.4. The fact that $X \subseteq U_r(N_r(X))$ in Theorem 4.2 does not rely on the assumption that $\mathrm{Cl}(X)$ is a convex subset of $\mathbb{R}^n$. The statement $X \subseteq U_r(N_r(X))$ holds for any set X. Refer to Figure 5 for an example.

The statements of Theorem 4.2 and Corollary 4.3 may seem far removed from doodles. It may help to recall that the boundary of the radius r neighborhood is our mathematical representation of the n-doodle. With this in mind, we see that Corollary 4.3 guarantees that we may return an n-doodle to its original shape provided that we know the original shape was convex. This, of course, also holds for the two dimensional doodles with which we started.

There are many related questions that arise from doodles which remain to be answered. For example, we might be curious if it is possible to reverse the n-doodle of any non-convex sets. This is in fact possible for any two distinct points in $\mathbb{R}^n$ with $n \geq 2$. Are there any other such examples? Certain non-convex shapes can also have reversible doodles if we place restrictions on the size of r. Consider the annulus in Figure 5 and fix r to be less than the radius of the inner circle. As long the inner circle does not get filled in by the radius r neighborhood, we may recover the original annulus. Perhaps another key observation to this example, is that the boundary of the annulus is a piecewise smooth curve. Can we generalize this example to any shape with a smooth boundary?

References

[1] Serge Lang, *Linear algebra*, third ed., Springer-Verlag New York, Inc., 1987.

[2] David C. Lay, *Linear algebra and its applications*, fourth ed., Pearson Education, Inc., 2012.

[3] James R. Munkres, *Topology*, second ed., Prentice Hall, Inc., 2000.

[4] Ravi Vakil, *The mathematics of doodling*, The American Mathematical Monthly **118** (2011), no. 2, 116–129.