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DETERMINING PROPERTIES OF METAL BY ANALYZING CHANGES IN IMPEDANCE

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Abstract. In certain situations it is useful to identify an unknown sample of metal without contact or visual inspection. We wish to do this by inducing a current in a coil and placing the sample in the resulting magnetic field. For the special case in which the sample is an infinite slab, we have a model that gives the change in impedance of the coil based on the properties of the sample. In this paper we analyze the inverse problem of finding the metal properties from impedance measurements over a wide range of frequencies.

1. Introduction

In technology such as metal detectors, it is useful to be able to identify a sample of metal without contact or visual inspection. Such technology already exists, but our ultimate goal is to find ways of improving the efficiency of current methods. These methods usually involve inducing an alternating current in a coil of wire and thus creating electric and magnetic fields. When an unknown metal is introduced close to the coil, these fields are distorted, producing a measurable change in the impedance of the coil. A model for the change in impedance at any given driving frequency is developed in [1] (pages 78-79); this impedance change depends on various material and geometric properties of the sample. Our goal is to solve the corresponding inverse problem, that is, to use measured impedance changes at various frequencies to deduce certain sample properties, specifically: lift-off (distance from the coil), conductivity, permeability, and thickness. In this paper we develop certain analytical results and numerical methods for obtaining these parameters.

2. Model

The general model we consider is a single coil over an infinite sheet with L layers of potentially different metals, as shown on the left in Figure 1; on the right is a 3D depiction of the setup with $L = 4$ layers, the bottom layer in this case being air. Note the Lth layer is assumed to extend to $z = -\infty$, and the layers are in fact assumed to be infinite slabs, that is, they extend to ∞ in the x and y directions. The parameters $\tilde{\mu}_t$ and σ_t denote the relative permeability and conductivity of the tth layer, while $-d_t$ is the z coordinate of the interface between layers t and $t+1$ (so that d_t is the depth of the interface below the sample top surface at $z = 0$.)

A general formula for the change in impedance (deviation from the impedance in free space or air) experienced by the coil at driving frequency ω is developed in [1], pages 78-79, and is

$$\Delta Z(\omega) = K \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1 - e^{-a\ell_2})^2 \frac{V_1}{U_1}\, da. \tag{1}$$

The various quantities that appear on the right in equation (1) are as follows. First, the quantity a is the variable of integration. The quantity ℓ_1 is the lift-off (the distance between

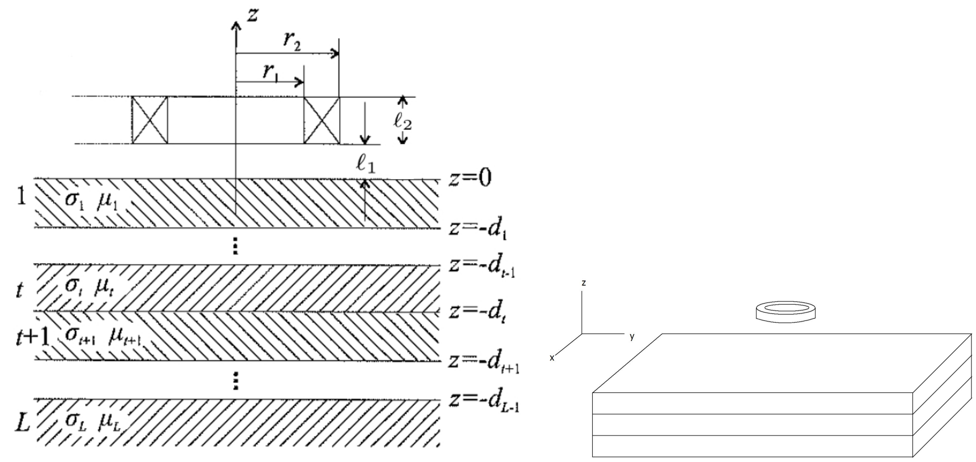

FIGURE 1. Diagrams of coil and layered structure.

the object and the coil) and ℓ_2 is the length of the coil, both as illustrated in Figure 1. The coefficient K is defined as

$$K = \frac{i\pi\omega\mu_0 N^2}{(r_2 - r_1)^2 \ell_2^2} \tag{2}$$

and contains ω, which is the frequency of the alternating current in the coil, while μ_0 denotes the permeability of free space, N denotes is the number of turns in the coil, and r_1 and r_2 denote the inner and outer radii of the coil, respectively (refer to Figure 1). The function I is defined by

$$I(a) = \int_{ar_1}^{ar_2} x J_1(x)\, dx$$

where $J_1(x)$ is the first order Bessel function of the first kind. Finally, the quantities U_1 and V_1, both functions of the integration variable a, are obtained from a recursively defined sequence of the form

$$U_t = \left(\frac{\lambda_{t-1}}{\tilde{\mu}_{t-1}} - \frac{\lambda_t}{\tilde{\mu}_t}\right) e^{-2\lambda_t d_t} V_{t+1} + \left(\frac{\lambda_{t-1}}{\tilde{\mu}_{t-1}} + \frac{\lambda_t}{\tilde{\mu}_t}\right) U_{t+1}$$

$$V_t = \left(\frac{\lambda_{t-1}}{\tilde{\mu}_{t-1}} + \frac{\lambda_t}{\tilde{\mu}_t}\right) e^{-2\lambda_t d_t} V_{t+1} + \left(\frac{\lambda_{t-1}}{\tilde{\mu}_{t-1}} - \frac{\lambda_t}{\tilde{\mu}_t}\right) U_{t+1}$$

The iteration ranges from $t = L - 1$ down to $t = 1$ where

$$\lambda_t = \sqrt{a^2 + i\omega\mu_0\tilde{\mu}_t\sigma_t},$$

but with provision that

$$\left(\frac{\lambda_{t-1}}{\tilde{\mu}_{t-1}}\right)_{t=1} = a$$

and with "initial" values

$$U_L = \frac{\lambda_{L-1}}{\tilde{\mu}_{L-1}} + \frac{\lambda_L}{\tilde{\mu}_L}$$

$$V_L = \frac{\lambda_{L-1}}{\tilde{\mu}_{L-1}} - \frac{\lambda_L}{\tilde{\mu}_L}.$$

In this paper we focus on the case with only one material layer of finite thickness in the structure, backed by air or free space. In this case $L = 2$, $\sigma_2 = 0$, $\tilde{\mu}_2 = 1$, and the formulae above become

$$V_1 = \left(a^2 - \frac{\lambda_1^2}{\tilde{\mu}_1^2}\right)(1 - e^{-2\lambda_1 d_1})$$
$$U_1 = \left(a + \frac{\lambda_1}{\tilde{\mu}_1}\right)^2 - \left(a - \frac{\lambda_1}{\tilde{\mu}_1}\right)^2 e^{-2\lambda_1 d_1}$$
$$\lambda_1 = \sqrt{a^2 + i\omega\mu_0\tilde{\mu}_1\sigma_1}.$$

Instead of solving the direct problem of calculating the change in impedance based on the properties of the metal, we approach the inverse problem of solving for $\tilde{\mu}_1$, σ_1, d_1, and ℓ_1 using measurements for ΔZ at various frequencies ω.

3. Analyticity of ΔZ and Uniqueness

Let us consider the quantity $\Delta(z)$ to be a function solely of Ω, with the parameters $\tilde{\mu}_1$, σ_1, d_1, and ℓ_1 fixed. We show below that $\Delta Z(\omega)$ a real-analytic function, that is, possesses a convergent power series expansion for each $\omega > 0$. From this we can deduce that knowledge of $\Delta Z(\omega)$ for any range of ω uniquely determines $\Delta Z(\omega)$ for all $\omega > 0$, and obtain partial uniqueness results for the inverse problem. The result also has implications for reconstruction of the unknown quantities.

The following theorem is proved in [2].

Theorem 1. *Suppose $\phi(z,t)$ is a continuous function of t, $A \leq t \leq B$, for fixed z and an analytic function of $z \in D$ for fixed t. Then*

$$f(z) = \int_A^B \phi(z,t)dt$$

is analytic in D.

We should note that the Theorem extends in a straightforward manner to the unbounded case in which, for example, $B = \infty$ if ϕ decays sufficiently rapidly at $t = \infty$. Specifically, if $\phi(z,t) = O(1/t^2)$ the transformation $t = \tan(u)$ can be used to reduce this unbounded case to that stated in the theorem.

It is not hard to see that $\Delta Z(\omega)$ may more generally be considered as a function of a complex variable ω with $\mathrm{Re}(\omega) > 0$. This is because then $\mathrm{Re}(\lambda_1) > 0$ and the integral defining $\Delta Z(\omega)$ remains convergent. Indeed, one can check that the integrand in (1) decays rapidly as a function of a, as $O(e^{-2a\ell_1})$. We can use Theorem 1 to show

Result 1. *The function $\Delta Z(\omega)$ is analytic as a function of ω for* $\mathrm{Re}(\omega) > 0$.

Proof. Let $\phi(\omega, a) = \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1 - e^{-a\ell_2})^2 \frac{V_1}{U_1}$. We begin by showing that $\phi(\omega, a)$ is continuous with respect to a for $0 \leq a < \infty$. Since ϕ is mainly composed of well known continuous functions, we consider only the continuity of $\frac{I^2(a)}{a^6}$ (in particular, near $a = 0$) and the possibility of U_1 being zero. Note that d_1 is positive.

The Bessel function J_1 can be represented as the following Taylor series with interval of convergence $-\infty < x < \infty$,

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}k!(k+1)!} x^{2k+1}.$$

Integrating term-by-term is permitted and after simplifying, we find that

$$\frac{I^2(a)}{a^6} = \sum_{k=0}^{\infty} \left(\frac{(-1)^k (r_2^{2k+3} - r_1^{2k+3})}{2^{2k+1} k!(k+1)!(2k+3)} \right)^2 a^{4k},$$

which we can see is a continuous function of a for $0 \leq a < \infty$.

Next, we consider the continuity of V_1/U_1 for $a > 0$. Note that it is clear that U_1 and V_1 are themselves continuous as functions of a. It's easy to check that $\lim_{a\to 0} V_1/U_1 = -1$, so that V_1/U_1 is continuous in some neighborhood of zero. To demonstrate the continuity of V_1/U_1 for $a > 0$ we will shows that $U_1 \neq 0$ when $a > 0$. To this end, suppose that $U_1 = 0$. This means that

$$\frac{(a + \frac{\lambda_1}{\tilde{\mu}_1})^2}{(a - \frac{\lambda_1}{\tilde{\mu}_1})^2} = e^{-2\lambda_1 d_1}. \tag{3}$$

We will show that the left side of (3) is always greater than or equal to 1, while the right side is always strictly less than 1, and so equation (3) can never be satisfied, and hence $U_1 \neq 0$ for any $a > 0$.

We first examine the right side of (3). We know $\lambda_1 = \sqrt{a^2 + iq}$ where q is some positive real number. A straightforward computation shows that $|\lambda_1| = (a^4+q^2)^{1/4} > 0$, and the point $\lambda_1^2 = a^2 + iq$ has complex argument $0 < \theta < \pi/2$. Then λ_1 has an argument $0 < \theta/2 < \pi/4$, and so lies in the first quadrant. Let us write $\lambda_1 = c_1 + ic_2$. Elementary geometry shows that

$$\begin{aligned} c_1 = |\lambda_1| \cos(\arg(\lambda_1)) &= (a^4 + q^2)^{1/4} \cos\left(\frac{\arctan(q/a^2)}{2}\right) \\ &= \frac{(a^4 + q^2)^{1/4}}{\sqrt{2}} \sqrt{1 + \frac{1}{\sqrt{q^2/a^4 + 1}}} \\ &= \frac{1}{2}(\sqrt{a^4 + q^2} + a^2) \\ &\geq \frac{1}{2}(a^2 + a^2) \\ &= a^2. \end{aligned}$$

where we make use of $\cos(\arctan(\alpha)) = 1/\sqrt{1 + \alpha^2}$ and the half angle formula $\cos(\beta/2) = \sqrt{(1 + \cos(\beta))/2}$. Thus $c_1 \geq a^2 > 0$. Observe:

$$\begin{aligned} |e^{-2\lambda_1 d_1}| &= |e^{-2c_1 d_1}||e^{-2ic_2 d_1}| \\ &= e^{-2c_1 d_1} \\ &\leq e^{-2ad_1} \\ &< 1. \end{aligned} \tag{4}$$

Now we examine the left side of (3). Observe that

$$\left| \left(a + \frac{\lambda_1}{\tilde{\mu}_1} \right)^2 \right| = \left(a + \frac{c_1}{\tilde{\mu}_1} \right)^2 + \frac{c_2^2}{\tilde{\mu}_1^2}$$

and

$$\left| \left(a - \frac{\lambda_1}{\tilde{\mu}_1} \right)^2 \right| = \left(a - \frac{c_1}{\tilde{\mu}_1} \right)^2 + \frac{c_2^2}{\tilde{\mu}_1^2}.$$

Since $c_1 > 0$ (and $a > 0, \tilde{\mu}_1 > 0$) we clearly have

$$\left| \frac{\left(a + \frac{\lambda_1}{\tilde{\mu}_1}\right)^2}{\left(a - \frac{\lambda_1}{\tilde{\mu}_1}\right)^2} \right| \geq 1. \tag{5}$$

Inequalities (4) and (5) clearly show that equation (3) can never be satisfied for any $a > 0$, that is, $U_1 \neq 0$ for $0 < a < \infty$. Therefore, ϕ is continuous with respect to a.

Since $\phi(\omega, a)$ is composed of products, sums, and quotients of functions that are analytic with respect to ω it is clear that $\phi(\omega, a)$ is analytic with respect to ω. In view of Theorem 1 the function $\Delta Z(\omega)$ is analytic for $0 < \omega < \infty$. □

Now we know that ΔZ is analytic in ω (and real-analytic if we confine ω to the real axis), we are justified in using its series expansion. Also, an real-analytic function is uniquely determined on its domain of definition if its values are known on an open interval within that domain, as per the following theorem.

Theorem 2. *Let f_1 and f_2 be analytic functions on domains Ω_1 and Ω_2, respectively, and suppose that the intersection $\Omega_1 \cap \Omega_2$ is not empty and that $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$. Then f_2 is called an analytic continuation of f_1 to Ω_2, and vice versa. Moreover, if it exists, the analytic continuation of f_1 to Ω_2 is unique.*[3]

From this we may immediately deduce

Theorem 3. *Let I be any open interval on the real line. The data $\Delta Z(\omega)$ for $\omega \in I$ uniquely determines $\Delta Z(\omega)$ for all $\omega > 0$.*

Of course the actual extrapolation of an analytic function to a larger domain is a very ill-posed problem, but Theorem 3 shows that if we have measurements of $\Delta Z(\omega)$ on any interval, the information therein contained is equivalent to that obtained from knowledge of $\Delta Z(\omega)$ for all positive ω.

4. Parameter Identification

Lift-off. In order to approximate the distance between the metal sample and coil, we examine the high frequency limit of $\frac{\Delta Z(\omega)}{K}$. When we look at this limit, we find, after a bit of analysis, that

$$\lim_{\omega \to \infty} \frac{\Delta Z(\omega)}{K} = \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1 - e^{-a\ell_2})^2(-1)\, da.$$

We can then approximate $\Delta Z(\omega)$ as $\Delta \tilde{Z}(\omega) = K \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1 - e^{-a\ell_2})^2(-1)\, da$ as long as ω is sufficiently large. This approximation is advantageous because it contains only one unknown parameter, the lift-off ℓ_1. Also, it's easy to see that for any fixed value of ω the quantity $\Delta \tilde{Z}(\omega)$ is a strictly increasing function of ℓ_1, and so ℓ_1 is uniquely determined by this limit. Thus we can use a bisection method to find a zero with respect to ℓ_1 of $f(x) = \Delta Z(\omega_0) - \Delta \tilde{Z}(\omega_0)$ where ω_0 is sufficiently large (on the order of 10^7 for the example we have considered).

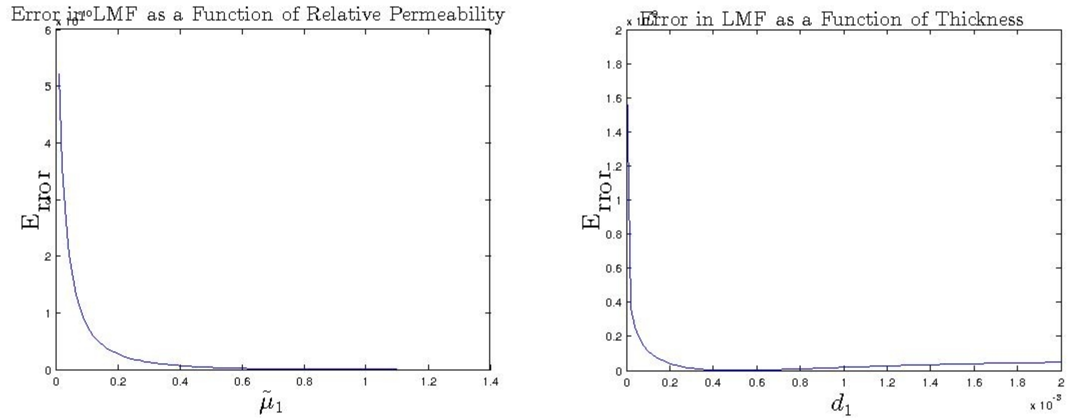

FIGURE 2. Error for sheet of silver

Conductivity. In order to find σ_1, we take the first two terms in the series expansion of $\frac{V_1}{U_1}$ with respect to large ω, and use this to approximate the integral. The first term is -1, which we already know from our approximation to find the lift-off. The second term in this expansion is $-\frac{2i\sqrt{i\mu_0\tilde{\mu}_1\sigma_1}a}{\mu_0\sigma_1}\frac{1}{\sqrt{\omega}}$. In what follows we assume that $\tilde{\mu}_1 \approx 1$ (thus excluding strongly magnetic materials). When this assumption holds,

$$\frac{\Delta Z(\omega)}{K} \approx \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1-e^{-a\ell_2})^2 \left(-1 - \frac{2i\sqrt{i\mu_0\sigma_1}a}{\mu_0\sigma_1}\frac{1}{\sqrt{\omega}}\right) da.$$

Using work from the previous section, we see that

$$\begin{aligned}\frac{\Delta Z(\omega)}{K} - \frac{\Delta\tilde{Z}}{K} &\approx \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1-e^{-a\ell_2})^2 \left(-\frac{2i\sqrt{i\mu_0\sigma_1}a}{\mu_0\sigma_1}\frac{1}{\sqrt{\omega}}\right) da \\ &= \frac{1}{\sigma_1^{1/2}} \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1-e^{-a\ell_2})^2 \left(-\frac{2i\sqrt{i\mu_0}a}{\mu_0}\frac{1}{\sqrt{\omega}}\right) da\end{aligned}$$

which now only contains the unknown parameter of σ_1. Also, we see clearly that when the limit is taken, σ_1 is uniquely determined because it is a constant with respect to the variable of integration. We again use a bisection method to accurately determine σ_1, assuming $\tilde{\mu}_1 \approx 1$ and ω is sufficiently large. In the case where our assumption that $\tilde{\mu}_1 \approx 1$ fails, we can use the algorithm in the following section to find $\sigma_1, d_1,$ and $\tilde{\mu}_1$ at the same time, but it becomes far less accurate when a third parameter is added.

Thickness and Relative Permeability. One method we use for finding the remaining parameters $d_1$and $\tilde{\mu}_1$ is the Levenberg-Marquardt-Fletcher (LMF) algorithm, which minimizes

$$\phi(d_1, \tilde{\mu}_1) = \sum_j |\Delta Z(d_1, \tilde{\mu}_1, \omega_j) - \Delta Z_j|^2$$

where each Z_j is the change in impedance measured at different frequencies ω_j. The algorithm typically converges to the correct answer (within some tolerance) with an initial guess that is sufficiently close to the actual parameters in question. Since these parameters have a fairly small range of values that they can take on ($0 < d_1 \leq 0.009$ and $0.8 \leq \tilde{\mu}_1 \leq 1.2$, approximately), finding an initial guess for which the algorithm converges is reasonably easy. This method is able to calculate the values of $d_1$and $\tilde{\mu}_1$ to within .01% of their true values (see Figure 2).

An alternate way to find thickness and relative permeability is to use a low ω approximation for $\Delta Z(\omega)$, taking the first two terms in the series expansion of $\frac{V_1}{U_1}$ and assuming that $\tilde{\mu}_1 \approx 1$. In this case

$$\frac{\Delta Z(\omega)}{K} \approx \int_0^\infty \frac{I^2(a)}{a^6} e^{-2a\ell_1}(1-e^{-a\ell_2})^2 \left(\frac{1}{4} \frac{\left(ie^{-2ad_1}\mu_0\sigma_1 - i\mu_0\sigma_1\right)}{a^2}\omega \right) da,$$

which only contains the unknown d_1. We may again use a bisection method to calculate d_1, assuming ω is small (on the order of 10^{-3} to 10^{-7}). We then have three out of four parameters and can use bisection on our original formula for ΔZ to find $\tilde{\mu}_1$. So far this method is slightly less accurate than the LMF approximation but shows promise.

5. Results

For an infinite, single-layer sheet of metal, we are able to determine the lift-off, thickness, relative permeability, and conductivity to a reasonable degree of accuracy. The parameters we use in the Table 1 are for a sheet of copper, and those in Table 2 are for a sheet of silver. In each case we use a range of frequencies from $\omega = 1$ to $\omega = 10^7$, corresponding to 0.159 Hz to 1.59×10^6 Hz. The parameters chosen for the coil are $\ell_2 = 2.3$mm, $r_1 = 5.35$mm, $r_2 = 7.7$mm, and $N = 100$. The initial guess given to the LMF algorithm was $d_1 = 5$mm and $\tilde{\mu}_1 = 1$. In Table 1, the lift-off and thickness are given in units of meters, conductivity is given in siemens, and relative permeability is unitless. The simulated data is noiseless.

ω	ΔZ
1	$5.6329\text{e}-09 - 1.0275\text{e}-07i$
10	$5.6302\text{e}-07 - 1.0342\text{e}-06i$
10^2	$5.4907\text{e}-05 - 1.6395\text{e}-05i$
10^3	$0.0032 - 0.0024i$
10^4	$0.0199 - 0.0821i$
10^5	$0.0709 - 0.9533i$
10^6	$0.2391 - 10.0640i$
10^7	$7.7172\text{e}-01 - 1.0232\text{e}+02i$

Parameter	Exact Value	Estimated Value	% Error
ℓ_1	0.0051	0.0051	0.44
d_1	0.0017	0.0017	0.01
$\tilde{\mu}_1$	0.999994	1.0000	0.001
σ_1	$5.96 \cdot 10^7$	$5.959 \cdot 10^7$	0.01

Table 1. Data and results for a copper sheet

6. Future Work

For future work, we would like to improve our numerical methods for the single layer case so that they are still accurate when relative permeability is not close to one. This might be accomplished by considering more terms in the expansion of V_1/U_1 at $\omega = \infty$. Additionally, more information concerning the thickness parameter d_1 or other parameters may be encoded

ω	ΔZ
1	$2.4562\text{e}-07-5.8407\text{e}-09i$
10	$2.2895\text{e}-05-4.6141\text{e}-06i$
10^2	$7.9415\text{e}-04-9.7006\text{e}-04i$
10^3	$0.0019-0.0177i$
10^4	$0.0057-0.1850i$
10^5	$0.0185-1.8910i$
10^6	$0.0591-19.0381i$
10^7	$1.8739\text{e}-01-1.9079\text{e}+02i$

Parameter	Exact Value	Estimated Value	% Error
ℓ_1	0.0034	0.0034	0.0740
d_1	0.0005	0.00049	0.0015
$\tilde{\mu}_1$	0.99998	1.0000	0.0020
σ_1	$6.2900 \cdot 10^9$	$6.2901 \cdot 10^9$	0.0017

Table 2. Data and results for a silver sheet

in an expansion of V_1/U_1 for low frequencies. More generally, it may also be possible to use our result that ΔZ is analytic to extrapolate ΔZ to the whole real line given data concerning ΔZ on an open interval. After this, we would also like to see if our methods will extend to the general case with multiple layers of varying types of metal. Finally, in order to make our work more applicable, we like to work with a model for a finite structure rather than an infinite sheet (a disc shape, for example). On the theoretical side of our problem, we would like to obtain more rigorous uniqueness results for the inverse problem, and perform a stability analysis for typical realistic configurations.

References

[1] MEI pre-print book
[2] Joseph Bak and Donald Newmann, *Complex Analysis*, 2nd ed., Springer, New York, 1997. p 224
[3] WolframAlpha.com Analytic Continuation