Constructing Probability Distributions Having a Unit Index of Dispersion

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CONSTRUCTING PROBABILITY DISTRIBUTIONS HAVING A UNIT INDEX OF DISPERSION

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Abstract. The index of dispersion of a probability distribution is defined to be the variance-to-mean ratio of the distribution. In this paper we formulate initial value problems involving second order nonlinear differential equations, the solutions of which are moment generating functions or cumulant generating functions of random variables having specified indices of dispersion. By solving these initial value problems we will derive relations between moment and cumulant generating functions of probability distributions and the indices of dispersion of these distributions. These relations are useful in constructing probability distributions having a given index of dispersion. We use these relations to construct several probability distributions having a unit index of dispersion. In particular, we demonstrate that the Poisson distribution arises very naturally as a solution to a differential equation.

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1 Introduction

A discrete random variable $X$ is said to follow a Poisson distribution with parameter $\lambda > 0$ if the probability mass function of $X$ is

$$p(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!}; & x = 0, 1, 2, 3, \ldots \\ 0; & \text{otherwise}. \end{cases}$$

Such a random variable enjoys the property that $E[X] = Var[X] = \lambda$. Thus, the index of dispersion, which is a normalized measure of dispersion defined as the ratio of the variance to the mean, for such a random variable is 1.

It is not difficult to see that the Poisson distribution with parameter $\lambda$ is obtained from the binomial distribution having parameters $n$ and $p$ when the sample size $n$ approaches infinity and the product $np$ approaches the positive constant $\lambda$. Thus, the Poisson distribution is very useful in modeling binomial experiments for which the sample size $n$ is large and the so-called probability of success $p$ is small. This is one reason why the Poisson distribution is important in applications and is widely used as a model for count data. Accordingly, we are often interested in testing the statistical hypothesis that a given set of data forms a sample from a population following a Poisson distribution, in such cases, we may make use of the fact that the index of dispersion of a Poisson distribution is 1 and let the null hypothesis be $H_0: \sigma^2/\mu = 1$. An estimator of the index of dispersion based on sample data can then serve as a test statistic for such a test.

It is not of interest to us in this paper to construct or comment on such statistical tests of the adequacy of the Poisson distribution as a model for an observed set of data, but assuming that a given statistical test supports the hypothesis that the population index of dispersion is 1, then we may ask, how “good” is this as an indicator that the population follows a Poisson distribution? For instance, if the Poisson distribution is the only probability distribution with a unit index of dispersion then a statistical test supporting the hypothesis that the population index of dispersion is 1 will immediately suggest that the population follows a Poisson distribution and no further investigation will be needed. Hence, we see that the following question, which we find to be of interest for its own sake, becomes relevant: Is the Poisson distribution the only probability distribution with mean equal to variance?

When stated in such generality, the answer to this question is no. In this paper, we shall demonstrate a method by which one can construct both discrete and continuous probability distributions having a unit index of dispersion. Such distributions will be reasonable models for data that seem to come from a population having a unit index of dispersion. To see why the answer to the above question is no, consider a normally distributed random variable having the density function

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right]; \quad -\infty < x < \infty$$
and simply set $\mu = \sigma^2$ to get a distribution with a unit index of dispersion. Having this answer, one might abandon the question stated above. However, this answer is not completely satisfactory since the normal distribution, unlike the Poisson distribution, is a probability distribution with two parameters, and the index of dispersion is usually a function of the parameters (if any) of the distribution, so when a distribution has more than one parameter, the parameters might be chosen to force the distribution to have a unit index of dispersion. Perhaps then, the question stated above should be modified by adding the requirement of having at most one parameter in the distribution.

Rather than looking for another distribution having one or no parameters and having a unit index of dispersion (which may not be a difficult task), we shall proceed in answering this question by translating the condition of having a unit index of dispersion into second order nonlinear differential equations, satisfying some initial conditions, the solutions of which will be moment generating functions or cumulant generating functions of random variables having the desired property. We may then recover the distribution functions of the random variables from their generating functions. By taking this approach, not only an answer for the question stated above will be given, but also we shall see that the Poisson distribution arises very naturally as a solution to a very simple differential equation. Furthermore, we will be able to derive some relations between the moment and cumulant generating functions of random variables and their variance-to-mean ratios.

We shall begin first by introducing the idea for the case of a discrete random variable, and construct a discrete probability distribution having a unit index of dispersion by solving a differential equation involving the moment generating function. In the third section of the paper, we will illustrate how the Poisson distribution arises very naturally as a solution to a very simple initial value problem involving the cumulant generating function. In the fourth section we will consider the case of continuous probability distributions.

Throughout this paper, whenever we refer to a random variable $X$, we shall assume that both the moment generating function $\varphi_X(t) = E[e^{itX}]$ and the cumulant generating function $\eta_X(t) = \log E[e^{itX}]$ of $X$ exist for $t$ in some neighborhood of 0. The expectation and variance of a random variable $X$ will henceforth be denoted by $E[X]$ and $Var[X]$, respectively. Finally, we shall assume that $E[X] \neq 0$ for any random variable $X$ mentioned in what follows so that the index of dispersion is always defined for any distribution considered here.

## 2 Discrete Probability Distributions

Let us begin by supposing that $X$ is a random variable for which the moment generating function $\varphi_X(t) = E[e^{itX}]$ exists for $t$ in some neighborhood of 0, it follows from the equations $E[X] = \varphi'_X(0)$ and $Var[X] = \varphi''_X(0) - [\varphi'_X(0)]^2$ that the index of dispersion of $X$, is

$$\frac{Var[X]}{E[X]} = \frac{\varphi''_X(0) - [\varphi'_X(0)]^2}{\varphi'_X(0)}.$$ 

We can write the last equation in the form of a differential equation as
\[
\frac{\varphi''_X(t) - [\varphi'_X(t)]^2}{\varphi'_X(t)} = H_\varphi(t), \tag{2.1}
\]

where \(H_\varphi\) is a function satisfying \(H_\varphi(0) = \text{Var}[X]/E[X]\). Moreover, since \(\varphi_X\) is a moment generating function, we have the initial conditions \(\varphi_X(0) = E[1] = 1\) and \(\varphi'_X(0) = E[X]\). Hence, having specified the function \(H_\varphi\) and a desired mean \(E[X] = \mu\), we may introduce the following initial value problem that facilitates the construction of a probability distribution with a given index of dispersion:

Find a solution \(\varphi_X\) to the differential equation

\[
\frac{\varphi''_X(t) - [\varphi'_X(t)]^2}{\varphi'_X(t)} = H_\varphi(t)
\]

on a given interval \(I\) that satisfies the initial conditions \(\varphi_X(0) = 1\) and \(\varphi'_X(0) = \mu\).

\text{(Initial Value Problem 2.1)}

In principle, the problem of constructing a probability distribution with a unit index of dispersion is now reduced to solving the boxed initial value problem above for a given choice of the function \(H_\varphi\) satisfying \(H_\varphi(0) = 1\), and then recovering the distribution of \(X\) from the moment generating function obtained. The simplest and most natural choice of \(H_\varphi\) is perhaps the constant function \(H_\varphi(t) = 1\). For such a choice, Equation 2.1 becomes

\[
\frac{\varphi''_X(t) - [\varphi'_X(t)]^2}{\varphi'_X(t)} = 1. \tag{2.2}
\]

The fact that the function \(\varphi_X\) does not appear in Equation 2.2 suggests reducing the order of the differential equation by setting \(\varphi'_X = \omega\). Indeed, this substitution transforms Equation 2.2 into a separable first order differential equation

\[
\frac{d\omega}{\omega(1 + \omega)} = dt.
\]

Using partial fraction decomposition and then integrating yields the solution \(\omega(t) = \frac{ke^t}{1-ke^t}\), where \(k\) is an integration constant. Let \(\mu\) denote the expectation of \(X\), from the initial condition \(\varphi'(0) = \omega(0) = \mu\), we have \(k = \mu/(1 + \mu)\). Hence, a solution to Equation 2.2 is given by

\[
\varphi_X(t) = \int \omega(t) \, dt = -\log \left[1 - \left(\frac{\mu}{1+\mu}\right)e^t\right] + c. \tag{2.3}
\]
Since our goal is merely to construct a probability distribution with a unit index of dispersion we may consider a particular solution corresponding to \( c = 0 \). Furthermore, since \( \varphi_X \) is assumed to be a moment generating function we must have \( \varphi_X(0) = E[1] = 1 \), that is, we must have \( \mu = e - 1 \). The particular solution we have chosen is favored since when \( c = 0 \) and \( \mu = e - 1 \) we have

\[
\varphi_X(t) = -\log[1 - (1 - e^{-1})e^t] = \sum_{x=1}^{\infty} \frac{(1 - e^{-1})^x}{x} e^{tx}. \tag{2.4}
\]

By comparing Equation 2.4 with the definition of the moment generating function for a discrete random variable \( X \), having support \( S \) and probability mass function \( p_X \):

\[
\varphi_X(t) := E[e^{tx}] = \sum_{x \in S} p_X(x) e^{tx},
\]

we see that if the random variable \( X \) is assumed to be discrete, then the probability mass function corresponding to the moment generating function in 2.4 is

\[
p_X(x) = \begin{cases} 
\frac{(1 - e^{-1})^x}{x} & \text{when } x = 1, 2, 3, \ldots \\
0 & \text{otherwise.}
\end{cases} \tag{2.5}
\]

We shall leave it to the reader to verify that the function \( p_X \) is indeed a probability mass function for a random variable \( X \) satisfying \( E[X] = Var[X] = e - 1 \).

**Remark** Having obtained \( \varphi_X(t) = -\log(1 - ke^t) \) one might introduce the normalizing constant \( -\log(1 - k) \) to obtain the probability mass function

\[
p_X(x) = \begin{cases} 
k^x & \text{when } x = 1, 2, 3, \ldots \\
-\log(1 - k)^x & \text{otherwise,}
\end{cases}
\]

where \( 0 < k < 1 \). However, this probability mass function will not have a unit index of dispersion since the function

\[
F(\varphi_X', \varphi_X'') = \frac{\varphi_X''(t) - [\varphi_X'(t)]^2}{\varphi_X'(t)}
\]

is not homogenous of degree zero, that is to say \( F(\lambda \varphi_X', \lambda \varphi_X'') \neq F(\varphi_X', \varphi_X'') \). Hence, if \( \varphi_X \) is a solution to Equation 2.2 then \( \lambda \varphi_X \) is not another solution whenever \( \lambda \neq 1 \).
Next, we proceed to generalize the arguments above by solving Initial Value Problem 2.1 for an arbitrary choice of the function \( H_\phi \). Let us suppose that \( X \) is a random variable such that \( E[X] = \mu \), where \( \mu \) is a nonzero constant. The left hand side of Equation 2.1 is just \( \frac{d}{dt} [\log|\varphi'_X(t)| - \varphi_X(t)] \) so we may write it as \( \frac{d}{dt} [\log|\varphi'_X(t)| - \varphi_X(t)] = H_\phi(t) \) and by integrating both sides we obtain

\[
\log|\varphi'_X(t)| - \varphi_X(t) = \int H_\phi(t) \, dt = I(t).
\]

Using the fact that \( I(0) = \log|\varphi'_X(0)| - \varphi_X(0) = \log|\mu| - 1 \) we may write the equation above as

\[
\log|\varphi'_X(t)| - \varphi_X(t) = \log|\mu| - 1 + \int_0^t H_\phi(u) \, du. \tag{2.6}
\]

Taking exponents in both sides of Equation 2.6 and then multiplying both sides by \(-1\) gives \(-\varphi'_X(t) \exp[-\varphi_X(t)] = -\mu e^{-1} \exp \left( \int_0^t H_\phi(u) \, du \right) \), or

\[
\frac{d}{dt} \exp[-\varphi_X(t)] = -\mu e^{-1} \exp \left( \int_0^t H_\phi(u) \, du \right).
\]

Integrating, using the fact that \( \varphi_X(0) = 1 \), taking logarithms and then multiplying both sides of the last expression by \(-1\) gives

\[
\varphi_X(t) = -\log \left[ e^{-1} + \int_0^t -\mu e^{-1} \exp \left( \int_0^\xi H_\phi(u) \, du \right) \, d\xi \right]. \tag{2.7}
\]

We summarize the main result of this section in the following theorem.

**Theorem 1** Let \( X \) be a random variable for which the moment generating function \( \varphi_X \) exists for \( t \) in some neighborhood of 0, \( Var[X] = \sigma^2 \), and \( E[X] = \mu \neq 0 \), then there exists a function \( H_\phi \) such that \( H_\phi(0) = \sigma^2/\mu \) and

\[
\varphi_X(t) = -\log \left[ e^{-1} + \int_0^t -\mu e^{-1} \exp \left( \int_0^\xi H_\phi(u) \, du \right) \, d\xi \right].
\]

Theorem 1 above gives us a relation between the moment generating function of \( X \) and its index of dispersion that is helpful in constructing probability distributions having a given index.
of dispersion provided we can recover the distribution from the moment generating function that we obtain from Equation 2.7. This requires a clever choice of the function \( H_\varphi \). For example, if we choose the constant function \( H_\varphi(t) = 1 \), as we did earlier to obtain a probability distribution with a unit index of dispersion, then we may easily obtain the probability mass function given by Equation 2.5, if we choose \( E[X] = e - 1 \), and under the assumption that \( X \) is a discrete random variable.

### 3 Constructing the Poisson Distribution

The real difficulty in using Theorem 1 to construct a distribution with a given index of dispersion lies in recovering the distribution from its moment generating function. This perhaps is mainly due to the fact that the expression of the index of dispersion in terms of the moment generating function is relatively complicated so the solution to Initial Value Problem 2.1 given by Equation 2.7 is also relatively complicated.

However, the formulae for the mean and variance of a random variable \( X \) in terms of the cumulant generating function of \( X \) are simpler than those written in terms of the moment generating function of \( X \). This suggests working with the cumulant generating function instead of the moment generating function. Indeed, the Poisson distribution arises very naturally when working with the cumulant generating function as we shall illustrate next.

If the cumulant generating function of \( X \) is \( \eta_X := \log E[e^{tx}] \), then \( E[X] = \eta'_X(0) \) and \( \text{Var}[X] = \eta''_X(0) \), thus the index of dispersion of \( X \) is given in terms of its cumulant generating function by \( \eta''_X(0)/\eta'_X(0) \). Thus, if \( X \) is a random variable for which the cumulant generating function \( \eta_X \) exists for \( t \) in some neighborhood of 0, then there exists a function \( H_\eta \) such that \( H_\eta(0) = \text{Var}[X]/E[X] \) and

\[
\frac{\eta''_X(t)}{\eta'_X(t)} = H_\eta(t). \tag{3.1}
\]

Furthermore, the differential equation above is associated with the initial conditions \( \eta_X(0) = \log E[1] = 0 \) and \( \eta'_X(0) = \mu \), where \( \mu = E[X] \). Thus, having specified the function \( H_\eta \) and a desired mean \( E[X] = \mu \) we may introduce the following initial value problem:

Find a solution \( \eta_X \) to the differential equation

\[
\frac{\eta''_X(t)}{\eta'_X(t)} = H_\eta(t)
\]

on a given interval \( I \) that satisfies the initial conditions \( \eta_X(0) = 0 \) and \( \eta'_X(0) = \mu \).

*(Initial Value Problem 3.1)*
To construct a probability distribution with a unit index of dispersion, let us begin, as we did earlier, by considering the simplest choice of $H_\eta$, namely the constant function $H_\eta(t) = 1$. For this choice of $H_\eta$, Equation 3.1 becomes

$$\frac{d^2}{dt^2} \log \eta_X(t) = 1 \quad \text{or} \quad \frac{d}{dt} \log \eta_X(t) = 1.$$ 

Integration of both sides yields $\log \eta_X(t) = t + c_1$. Setting $t = 0$ in the last equation gives $\log \mu = c_1$, where $\mu = E[X]$. Thus we have $\log \eta_X(t) = t + \log \mu$ which upon taking exponents and integrating gives $\eta_X(t) = \mu e^t + c_2$, but since $\eta_X$ is a cumulant generating function we must have $\eta_X(0) = \log \phi_X(0) = 0$ and so $c_2 = -\mu$. Hence, $\eta_X(t) = \mu e^t - \mu$ which we may write in terms of the moment generating function as $M_X(t) = e^{-\mu } e^{\mu t}$ (the reader may recognize this as the moment generating function of a Poisson random variable with parameter $\mu$). Assuming the random variable $X$ is discrete we may use the Taylor series expansion to get

$$\phi_X(t) = e^{-\mu } e^{\mu t} = \sum_{x=0}^{\infty} \frac{e^{-\mu } \mu^x}{x!} e^{tx}.$$ 

It is now apparent from Equation 3.2 that the probability mass function of $X$ is

$$p(x) = \begin{cases} \frac{e^{-\mu } \mu^x}{x!}, & x = 0, 1, 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases}$$ 

Because the probability mass function cannot take negative values, we are bound to have $\mu > 0$. Thus, the random variable $X$ here is a Poisson random variable with parameter $\mu > 0$. The discussion above can be easily generalized to obtain the following theorem.

**Theorem 2** Let $X$ be a random variable for which the cumulant generating function $\eta_X$ exists for $t$ in some neighborhood of 0, $Var[X] = \sigma^2$, and $E[X] = \mu \neq 0$, then there exists a function $H_\eta$ such that $H_\eta(0) = \sigma^2 / \mu$ and

$$\eta_X(t) = \int_0^t \exp \left( \int_0^\xi H_\eta(u) \, du \right) \, d\xi.$$ 

The proof of Theorem 2, the details of which we omit, constitutes of solving Initial Value Problem 3.1.

## 4 Continuous Probability Distribution

We have assumed in the first section of this paper that $\phi_X(t) = -\log(1 - ke^t)$ is the moment generating function of a discrete random variable and then we have obtained the probability mass
function in Equation 2.5 under this assumption. If for some choice of $H_\varphi$, we suppose that the moment generating function $\varphi_X$ is that of a continuous random variable $X$, then we need to find the probability density function $f$ in the expression

$$\varphi_X(t) := \int_{-\infty}^{\infty} e^{tx} f(x) \, dx,$$

where the function $\varphi_X$ is given by Equation 2.7, and this task may be difficult. However, to take continuous random variables in consideration, we may take advantage of the widely tabulated Laplace transforms. Let us begin by assuming that we have a nonnegative continuous random variable $X$. Since $X \geq 0$, the moment generating function of $X$ is

$$\varphi_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx = \int_{0}^{\infty} e^{tx} f(x) \, dx. \quad (4.1)$$

Under the transformation $t \rightarrow -t$ Equation 4.1 becomes

$$F(t) = \varphi_X(-t) = \int_{0}^{\infty} e^{-tx} f(x) \, dx, \quad (4.2)$$

and we recognize this as the Laplace transform $\mathcal{L}[f]$ of $f$. Moreover, under this transformation we have

$$-F'(t) = -\frac{d}{dt} \varphi_X(-t) = \varphi'_X(-t), \quad \text{and}$$

$$F''(t) - [F'(t)]^2 = \frac{d^2}{dt^2} \varphi_X(-t) - \left[\frac{d}{dt} \varphi_X(-t) \right]^2$$

$$= \varphi''_X(-t) - [\varphi_X(-t)]^2.$$

Thus the expectation and variance of $X$ are given in terms of $F$ by

$$E[X] = -F'(0),$$

$$Var[X] = F''(0) - [F'(0)]^2,$$

and so, requiring a specific value of the index of dispersion translates to the differential equation

$$\frac{F''(t) - [F'(t)]^2}{F'(t)} = -Q_\varphi(t), \quad (4.3)$$
where $Q_\varphi$ is a function satisfying $Q_\varphi(0) = \text{Var}[X]/\text{E}[X]$.

Similar to what we did earlier, we introduce the following initial value problem for a given choice of the function $Q_\varphi$ and a given choice of $\mu = \text{E}[X]$.

Find a solution $F$ to the differential equation

$$\frac{F''(t) - [F'(t)]^2}{F'(t)} = -Q_\varphi(t)$$

on a given interval $I$ that satisfies the initial conditions $F(0) = 1$ and $F'(0) = -\mu$.

*(Initial Value Problem 4.3)*

If $F$ is a solution to Initial Value Problem 4.3 then the inverse Laplace transform will be the function $f$ appearing in Equation 4.2, that is $f = \mathcal{L}^{-1}[F]$. Equation 4.3 is equivalent to Equation 2.1 with $F = \varphi_X$ and $H_\varphi = -Q_\varphi$ and so, it follows from the fact that the function given by Equation 2.7 is a solution to Initial Value Problem 2.1, that a solution to Initial Value Problem 4.3 is

$$F(t) = -\log \left[ e^{-1} + \int_0^t \mu e^{-1} \exp \left( \int_0^\xi -Q_\varphi(u) du \right) d\xi \right]. \tag{4.4}$$

Hence, the probability density function of $X$ is given by

$$f(x) = \mathcal{L}^{-1} \left\{ -\log \left[ e^{-1} + \int_0^t \mu e^{-1} \exp \left( \int_0^\xi -Q_\varphi(u) du \right) d\xi \right] \right\}. \tag{4.5}$$

Let us now construct a probability density function $f$ of a continuous random variable having a unit index of dispersion. To obtain a simple expression for $F$ given by Equation 4.4, so that it would be easy to find the inverse Laplace transform of $F$, we must choose the function $Q_\varphi$ carefully. After some attempts, we realize that we must choose the function $Q_\varphi$ so that its integral will involve the logarithm function. If we choose $Q_\varphi(u) = \frac{2}{u+1} - \frac{1}{(u+1)^2}$, (observe that $Q_\varphi(0) = 1$) then, as the reader may verify, we have

$$e^{-1} + \int_0^t \mu e^{-1} \exp \left( \int_0^\xi -Q_\varphi(u) du \right) d\xi = e^{-1} - \mu e^{-1} + \mu e^{-\frac{1}{t+1}}.$$
Since our goal is merely to construct a probability distribution, we may set $\mu = 1$ so that $F(t) = -\log\left(e^{-\frac{1}{1+t}}\right)$. From Equation 4.5 we have

$$f(x) = \mathcal{L}^{-1}\left[-\log\left(e^{-\frac{1}{1+t}}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{1+t}\right] = e^{-x}.$$

Thus, the probability distribution of $X$ is

$$f(x) = \begin{cases} e^{-x}; & x \geq 0 \\ 0; & x < 0. \end{cases} \quad (4.6)$$

We recognize the distribution in Equation 4.6 as being an exponential distribution with parameter 1.

## 5 Conclusion

We have formulated initial value problems, the solutions of which are moment generating functions or cumulant generating functions of random variables having specified indices of dispersion. By solving these initial value problems we were able to derive relations between moment and cumulant generating functions of probability distributions and the indices of dispersion of these distributions (see Theorem 1, Theorem 2 and the discussion in Section 4), and to construct probability distributions with a unit index of dispersion.

In Section 2, we translated the condition of having a unit index of dispersion into an initial value problem involving the moment generating function, and as a result we were able to construct the probability mass function

$$p_X(x) = \begin{cases} \frac{(1 - e^{-1})x}{x}; & \text{when } x = 1, 2, 3, \ldots \\ 0; & \text{otherwise}, \end{cases}$$

by choosing the function $H_\varphi$ (appearing in Initial Value Problem 2.1) to be the constant function $H_\varphi(t) = 1$. Thus, this probability mass function arises very naturally when working with the moment generating function. The transition from working with the moment generating function to working with the cumulant generating function in Section 3 enabled us to recover the Poisson distribution in a similar fashion. However, we know from our discussion in Section 2, that for some choice of the function $H_\varphi$ satisfying $H_\varphi(0) = 1$, the solution to Initial Value Problem 2.1 will be the moment generating function of the Poisson distribution. The reader may attempt to construct several other distributions by taking different choices of the function $H_\varphi$ but the
Poisson distribution will be difficult to obtain in this manner. We may find the function $H_{\varphi}$ corresponding to the Poisson distribution simply by substituting the moment generating function of the Poisson distribution $\varphi_x(t) = e^{-\lambda} e^{\lambda e^t}$ in the expression

$$\frac{\varphi''_x(t) - [\varphi'_x(t)]^2}{\varphi'_x(t)},$$

to get

$$H_{\varphi}(t) = 1 + \lambda e^t - \lambda e^{-\lambda e^t} e^{\lambda e^t}. \quad (5.1)$$

If we had no prior knowledge of the Poisson distribution, it would have been very difficult to construct it simply by solving Initial Value Problem 2.1, since this requires choosing $H_{\varphi}$ to be the function in 5.1 which does not seem to be a natural choice. However, working with the cumulant generating function instead of the moment generating function makes the construction of the Poisson distribution very natural as illustrated in Section 3. Perhaps then, working with other generating functions such as the factorial moment generating function will facilitate the construction of probability distributions having a unit index of dispersion that are difficult to construct when our considerations are limited to the moment and cumulant generating functions only.

Furthermore, investigating the adequacy of such distributions as models for data sets that come from populations having unit indices of dispersion would be an interesting task. Indeed, when empirical evidence or information obtained from sample data suggests the inadequacy of the Poisson distribution as a model; such distributions may be considered as possible alternatives.

References

