Rose-Hulman Undergraduate Mathematics Journal

ANALLAGMATIC CURVES AND INVERSION ABOUT THE UNIT HYPERBOLA

Stephanie Neas[a]

Volume 18, No. 1, Spring 2017

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
mathjournal@rose-hulman.edu
scholar.rose-hulman.edu/rhumj

[a]University of Wisconsin - Stout

Rose-Hulman Undergraduate Mathematics Journal
Volume 18, No. 1, Spring 2017

Anallagmatic Curves and Inversion About the Unit Hyperbola

Stephanie Neas

Abstract. In this paper we investigate inversion about the unit circle from a complex perspective. Using complex rational functions we develop methods to construct curves which are self-inverse (anallagmatic). These methods are then translated to the split-complex numbers to investigate the theory of inversion about the unit hyperbola. The analog of the complex analytic techniques allow for the construction and study of anallagmatic curves about the unit hyperbola.

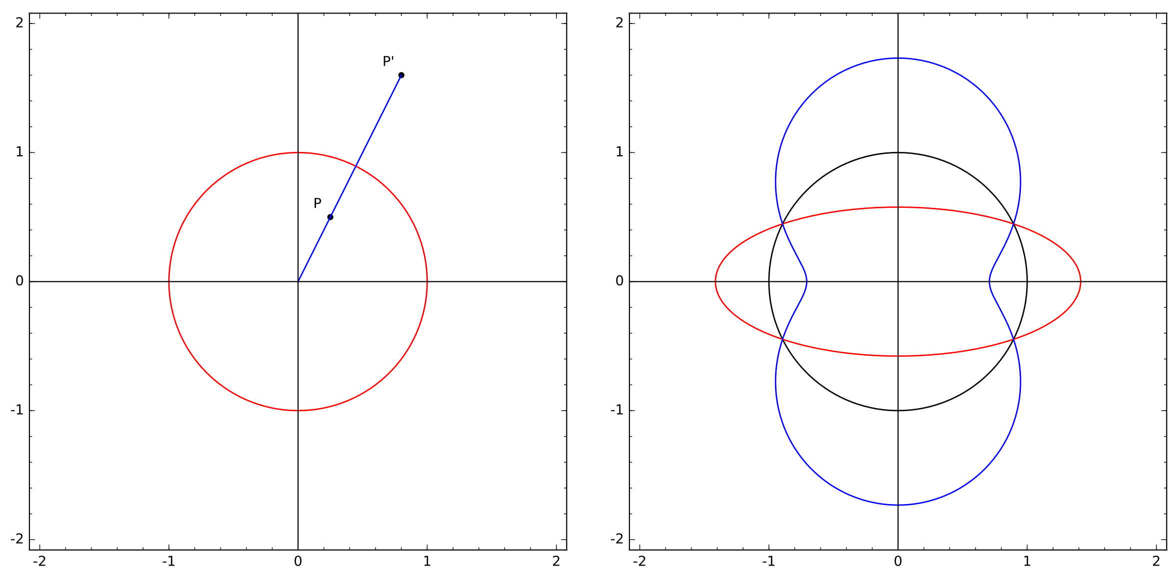
Acknowledgements: This research was supported by the UW-Stout Foundation and the McNair Scholars Program.

1 Introduction

Inversion about a circle has been studied in classical geometry ([2], [5]) and in the field of harmonic analysis in terms of the Kelvin transform. With respect to a circle, the inverse of a point is defined as a point which lies on the same ray as the original point and has reciprocal magnitude to that point. Also, the inverse of a curve is defined by taking the inverse of each point on the curve. See Figure 1 for an example of each. Any curve that is its own inverse is called an *anallagmatic*, or self-inverse, curve. See Figure 2 for an example of an anallagmatic curve.

In addition, inversion about other curves has also been studied. Childress [4] gives a definition of inversion about an arbitrary conic section. Recently Ramírez [7] has given a more thorough treatment of inversion about ellipses as well as a generalization of the Pappus Chain Theorem.

In this paper we explore inversion about the unit circle and the unit hyperbola, and through complex analysis establish methods of constructing anallagmatic curves about both. In Section 2.1 we will review the definitions and formulas for inversion about the unit circle. These concepts are translated to the case of inversion about the unit hyperbola in Section 3.1.



(a) A point P and the inverse point P'. (b) An ellipse and the inverse curve drawn in blue.

Figure 1: Figures showing an inverse point and an inverse curve about the unit circle.

For both cases we define and discuss curves that are anallagmatic about the unit circle (Section 2.2) and unit hyperbola (Section 3.1).

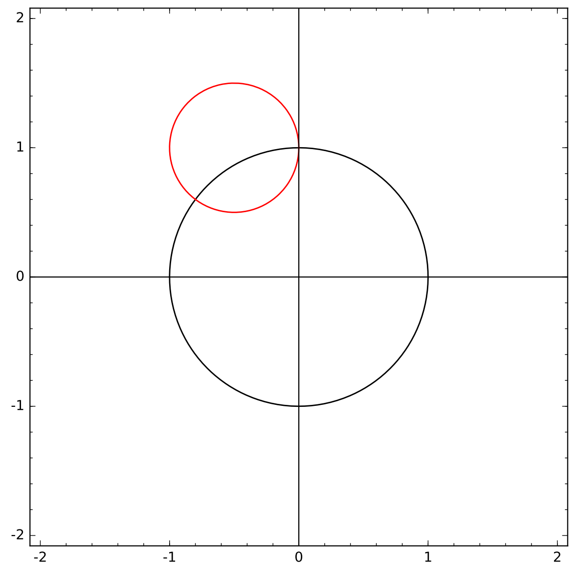


Figure 2: An example of an anallagmatic curve.

Inversion about the unit circle is well suited for study via the complex numbers. A treatment from this perspective is provided in Section 2.3. Using arguments analogous to the complex analytic case we develop an approach for investigating inversion about the unit hyperbola through the split-complex numbers introduced in Section 3.2.

In both the complex and split-complex cases rational transformations are constructed to reduce the analysis of anallagmatic curves to the study of curves which are symmetric about the real axis. In Section 2.4 we provide descriptions of such transformations for the complex plane and prove that the Cayley transform is one example. In Section 3.3 the analogous theory is established for the split-complex plane. The equations for the various curves throughout the paper are provided in Section 4. The graphs of the curves were created in SageMath. We now give a brief review of inversion about the unit circle from an algebraic and complex perspective.

2 Inversion with Respect to the Unit Circle

2.1 Inversion About the Unit Circle

In order to motivate our constructions on inversion about a unit hyperbola, we first review the basics of inversion about the unit circle by formalizing the definitions mentioned in the introduction. It is possible to define inversion about circles with radii other than 1 as well as circles not centered at the origin. However, we will focus strictly on the case of the unit circle centered at the origin. This particular restriction will lend itself well to using complex numbers in subsequent sections.

All constructions and definitions will take place in the plane which we will denote by either $\mathbb{R}^2$ or $\mathbb{C}$. We will denote the origin, $(0,0)$, by the letter O, or in the complex case by the number 0.

Definition 2.1. The point P' is said to be the *inverse* of the point P with respect to the unit circle if P' lies on the ray from O through P and $OP \cdot OP' = 1$.

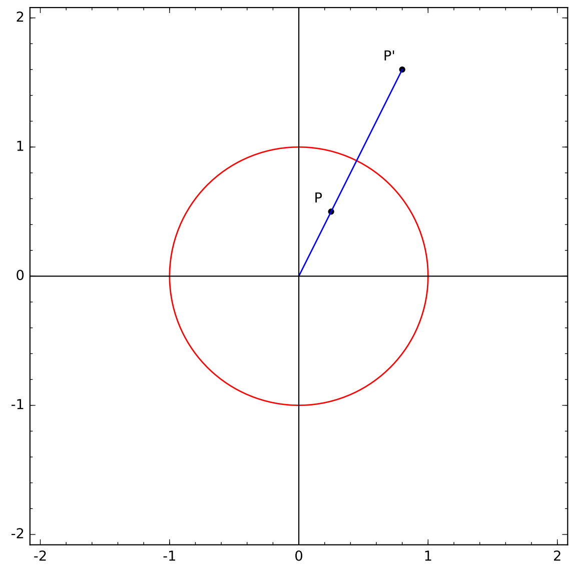


Figure 3: A point P located at (1/4, 1/2), and its inverse P' located at (4/5, 8/5).

By the symmetric nature of this definition, the inverse of the point P' is the original point P. In other words, the inverse of the inverse is the original point. With the exception of O, which has no inverse, every point has a uniquely defined inverse. Therefore, we will unambiguously refer to *the* inverse of a point.

As inversion is well defined on the complement of the origin we define the inversion function

$$\psi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$$

by $\psi(P) = P'$; that is, ψ takes any point to its inverse.

Proposition 2.2. *In Cartesian coordinates*

$$\psi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \tag{2.1}$$

and in polar coordinates $\psi(r,\theta) = (1/r, \theta)$.

Proof. In the case of polar coordinates it is immediate that the points (r,θ) and $(1/r,\theta)$ are on the same ray through the origin. The product of their magnitudes is $|r| \cdot |1/r| = 1$ and therefore these points are inverse to each other. Converting from polar to Cartesian we have that $(1/r, \theta)$ becomes $(\cos(\theta)/r, \sin(\theta)/r)$. However,

$$\begin{aligned}(\cos(\theta)/r, \sin(\theta)/r) &= \left(\frac{r\cos(\theta)}{r^2}, \frac{r\sin(\theta)}{r^2}\right) \\ &= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).\end{aligned}$$

□

In addition to taking the inverse of a point, one can take the inverse of a subset of the plane. In practice, the only subsets other than points that we will be considering are curves, but we will define the inversion of a set in general.

Definition 2.3. For any subset $C \subset \mathbb{R}^2 \setminus \{(0,0)\}$ of the plane we define the *inverse* of C to be

$$C' = \{\psi(P) : P \in C\}.$$

If a set $C \subset \mathbb{R}^2$ contains $(0,0)$, we will still refer to the inverse C'. In this case we simply omit the point $(0,0)$ from C before taking the inverse.

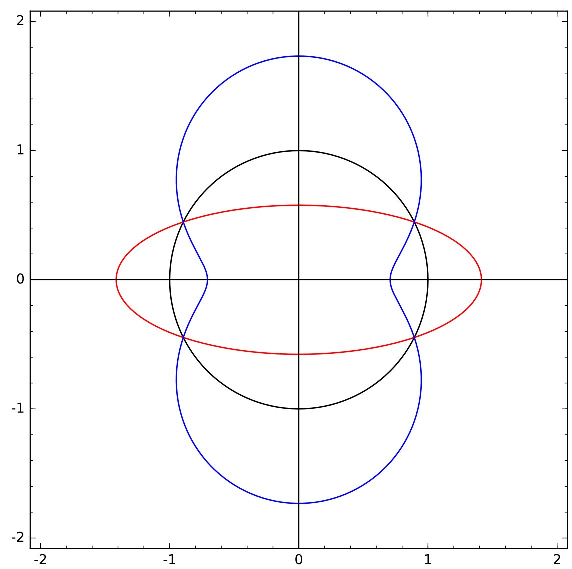


Figure 4: A red ellipse defined by the equation $\frac{1}{2}x^2 + 3y^2 - 1 = 0$, and its inverse is drawn in blue.

The two types of curves that will be of interest are parametrically defined and implicitly defined curves. If a curve C is defined by a parametric equation $(x(t), y(t))$, then the inverse curve is defined by

$$\psi(x(t), y(t)) = \left(\frac{x(t)}{x(t)^2+y(t)^2}, \frac{y(t)}{x(t)^2+y(t)^2}\right).$$

On the other hand if C is defined implicitly by $f(x,y)=0$, then C' is defined implicitly by

$$f\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = 0.$$

Example 2.4. The ellipse in Figure 4 can be defined by the parametric equation $\left(\sqrt{2}\cos(t), \frac{\sqrt{3}}{3}\sin(t)\right)$. The equation of its inverse curve is given by

$$\begin{aligned}\psi\left(\sqrt{2}\cos(t), \frac{\sqrt{3}}{3}\sin(t)\right) &= \left(\frac{\sqrt{2}\cos(t)}{\left(\sqrt{2}\cos(t)\right)^2 + \left(\frac{\sqrt{3}}{3}\sin(t)\right)^2}, \frac{\frac{\sqrt{3}}{3}\sin(t)}{\left(\sqrt{2}\cos(t)\right)^2 + \left(\frac{\sqrt{3}}{3}\sin(t)\right)^2}\right) \\ &= \left(\frac{3\sqrt{2}\cos(t)}{6\cos(t)^2+\sin(t)^2}, \frac{\sqrt{3}\sin(t)}{6\cos(t)^2+\sin(t)^2}\right).\end{aligned}$$

Alternatively, the ellipse can also be defined by the implicit equation $\frac{x^2}{2}+3y^2-1=0$. Its inverse can then be described by the equation

$$\frac{1}{2}\left(\frac{x}{x^2+y^2}\right)^2 + 3\left(\frac{y}{x^2+y^2}\right)^2 - 1 = 0$$

which can be rewritten as

$$\frac{-(2y^4+4x^2y^2-6y^2+2x^4-x^2)}{2(x^2+y^2)^2} = 0.$$

Note that $(0,0)$ is a zero of the numerator in the above equation, however, as it is not in the domain of this rational function, $(0,0)$ is not on the curve.

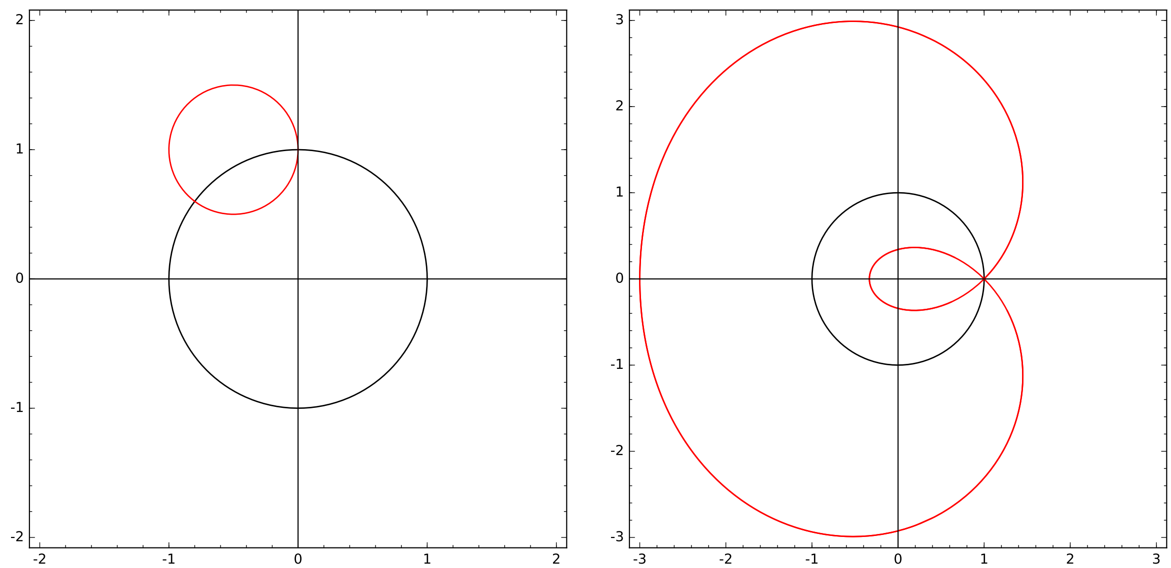
2.2 Anallagmatic Curves

Any parametrically or implicitly defined curve has an inverse which can be computed algebraically as described in the previous section. On the other hand we can consider the question of which curves are their own inverse. Before doing so we introduce the following terminology.

Definition 2.5. An *anallagmatic curve* is a curve whose inverse with respect to the unit circle is itself.

Alternatively, this means that a curve C is anallagmatic if and only if for every point $P \in C$ we also have that the point $P' \in C$. For two examples of anallagmatic curves see Figure 5.

For either parametrically or implicitly defined curves it is nontrivial to identify whether the defining equation gives an anallagmatic curve or not. It can also be challenging to construct such curves. To remedy this, we approach the matter from a complex analytic perspective.



(a) An anallagmatic circle. (b) A more complicated anallagmatic curve.

Figure 5: Two anallagmatic curves. The black circle is the unit circle in each picture.

2.3 Inversion From a Complex Perspective

This section is intended to provide a review of basic constructions from complex analysis relating to inversion of points. While inversion about the unit circle via complex numbers is commonly covered in complex analysis textbooks (see Chapter 3 of the text by Needham [6]), it is included here for the reader's convenience and to motivate the split-complex approach taken in Section 3.2. Additionally, we establish new results relating anallagmatic curves and complex rational functions that are real on the unit circle.

Proposition 2.6. *If* $z \in \mathbb{C} \setminus \{0\}$*, then the inverse of the point* z *about the unit circle is given by* $1/\bar{z}$*.*

Proof. If we write z in polar form, $z = re^{i\theta}$, then

$$\frac{1}{\bar{z}} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}.$$

Which, by Proposition 2.2, is the inverse of z in polar form. □

For a nonzero complex number z we will denote the inverse about the unit circle by z^*. When referring to an inverse we will always mean the inverse of the point about the unit circle (the conjugate reciprocal) and not simply the multiplicative inverse (reciprocal).

In the complex plane, the level sets of the real part of a complex meromorphic function describe a family of curves. We now give a classification of complex rational functions such that all level sets of the real part define anallagmatic curves.

Theorem 2.7. *Let $f(z)$ be a nonzero complex rational function. Then the curve defined by $\Re\{f(z)\} - d = 0$ is anallagmatic for every real number d if and only if $f(z) - \frac{i}{2}c$ is real valued on the unit circle for some real number c.*

Proof. Throughout this proof we will only consider points in the domain of $f(z)$. Since $f(z)$ is a rational function, this is a finite set of points being omitted. Suppose now that $\Re\{f(z)\}-d = 0$ defines an anallagmatic curve for every $d \in \mathbb{R}$. If $z_0 \in \mathbb{C}\backslash\{0\}$ is in the domain of $f(z)$, then by hypothesis $\Re\{f(z)\} - \Re\{f(z_0)\} = 0$ defines an anallagmatic curve. Because z_0 is on this curve z_0^* is as well. In particular, we must have $\Re\{f(z_0^*)\} = \Re\{f(z_0)\}$ for all $z_0 \neq 0$ in the domain of $f(z)$. As $\Re\{f(z)\}$ is real valued, it follows that $\Re\{f(z)\} = \Re\{\overline{f(z^*)}\}$ when defined. Consequently,

$$\Re\{f(z) - \overline{f(z^*)}\} = 0. \tag{2.2}$$

Since $\overline{f(z^*)}$ is also a rational function in z, the above difference is a complex rational function. By the Cauchy-Riemann equations we must have

$$\Im\{f(z) - \overline{f(z^*)}\} = c, \tag{2.3}$$

where c is some real constant.

For convenience we define $g(z) = f(z) - \frac{i}{2}c$. It now suffices to prove that $g(z)$ is real valued on the unit circle. Because $g(z)$ and $f(z)$ have equal real parts, it follows from Equation 2.2 that

$$\Re\{g(z) - \overline{g(z^*)}\} = 0.$$

Similarly by Equation 2.3 we have

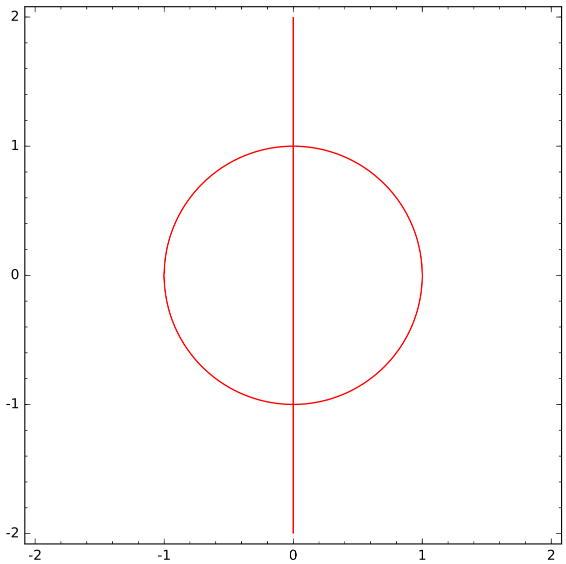
$$\begin{aligned}\Im\{g(z) - \overline{g(z^*)}\} &= \Im\{f(z) - \frac{i}{2}c - (\overline{f(z^*)} + \frac{i}{2}c)\} \\ &= \Im\{f(z) - \overline{f(z^*)}\} - c \\ &= 0.\end{aligned}$$

As the real and imaginary parts of $g(z)$ and $\overline{g(z^*)}$ are equal

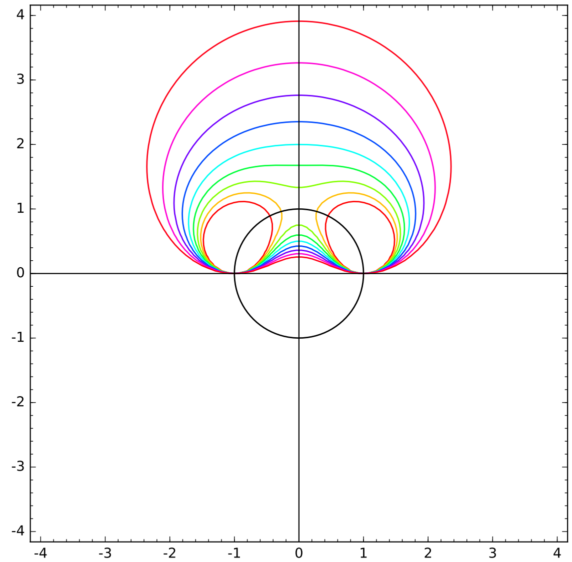
$$g(z) = \overline{g(z^*)} \tag{2.4}$$

for all z in the domain. Note that if z_0 is on the unit circle, then $\overline{z_0^*} = z_0$. In particular, if we substitute this equality into Equation 2.4 we get $g(z_0) = \overline{g(z_0)}$. Thus, $g(z)$ is real valued on the unit circle when defined.

For the other direction, we now suppose that $f(z) - \frac{i}{2}c$ is a real valued function on the unit circle for some real number c. As the conclusion of the theorem only concerns the real part of $f(z)$, and the $\frac{i}{2}c$ term only changes the imaginary part, we will assume that $f(z)$



(a) The curves defined by $\Im\{f(z)\} = 0$, which are the union of the unit circle and the imaginary axis.



(b) A few members of the family of curves of the form $\Re\{f(z)\} - d = 0$, which are anallagmatic with respect to the unit circle.

Figure 6: Level sets for the imaginary and real parts of the function $f(z) = \dfrac{2z^2 - 5iz - 2}{2(z^2 - 1)}$.

was real on the unit circle in the first place. Let z_0 be on the unit circle and in the domain of $f(z)$. Then we have

$$f(z_0) = f(z_0^*) = \overline{f(z_0^*)}.$$

Since $f(z) - \overline{f(z^*)}$ is a complex rational function, it either has finitely many zeroes or it is identically 0. However, it vanishes at infinitely many points on the unit circle, therefore $f(z) = \overline{f(z^*)}$ when defined. Let d be an arbitrary real number and suppose that $z_1 \neq 0$ is such that $\Re\{f(z_1)\} - d = 0$. Then we also have that

$$\Re\{\overline{f(z_1^*)}\} - d = 0.$$

From which it follows that

$$\Re\{f(z_1^*)\} - d = 0.$$

Consequently, whenever z_1 is in the level set described by $\Re\{f(z_1)\} - d = 0$, so is z_1^*, and the curves defined by $\Re\{f(z)\} - d = 0$ must be anallagmatic. □

See Figure 6 for an example of the above theorem. While Theorem 2.7 provides one method for constructing anallagmatic curves, it does not generate all of them. In the next section we will provide an additional technique for constructing arbitrary anallagmatic curves.

2.4 Rational Function Transformations

To construct and analyze general anallagmatic curves we translate the property of points being inverse to each other about the unit circle to points being symmetric about the real axis.

Theorem 2.8. *Let $f(z)$ be a complex rational function of the form*

$$f(z) = a \prod_{k=1}^{n} \left(\frac{z - \alpha_k}{z - \bar{\alpha}_k} \right),$$

where the $\alpha_k \in \mathbb{C}$ and $a \in \{1, -1\}$. Then for any point $z_0 \in \mathbb{C}$ where $f(z_0)^*$ is defined, $f(z_0)^* = f(\bar{z}_0)$.

Proof. Let f and z_0 be as in the hypothesis. Then

$$f(z_0)^* = \overline{\left(\frac{1}{f(z_0)} \right)} = \overline{\frac{1}{a} \prod_{k=1}^{n} \left(\frac{z_0 - \bar{\alpha}_k}{z_0 - \alpha_k} \right)}.$$

However, since $\overline{1/a} = a$, and by distributive properties of the conjugation operator, we have

$$f(z_0)^* = a \prod_{k=1}^{n} \left(\frac{\bar{z}_0 - \alpha_k}{\bar{z}_0 - \bar{\alpha}_k} \right) = f(\bar{z}_0).$$

□

A consequence of this theorem is that one can construct an anallagmatic curve by starting with any curve which is symmetric about the real axis and then applying a rational function of the above form.

Lemma 2.9. *Let $f(z)$ be as in Theorem 2.8. Then the image of any real number in the domain of $f(z)$ is on the unit circle.*

Proof. Let $x \in \mathbb{R}$ be in the domain of f. Then

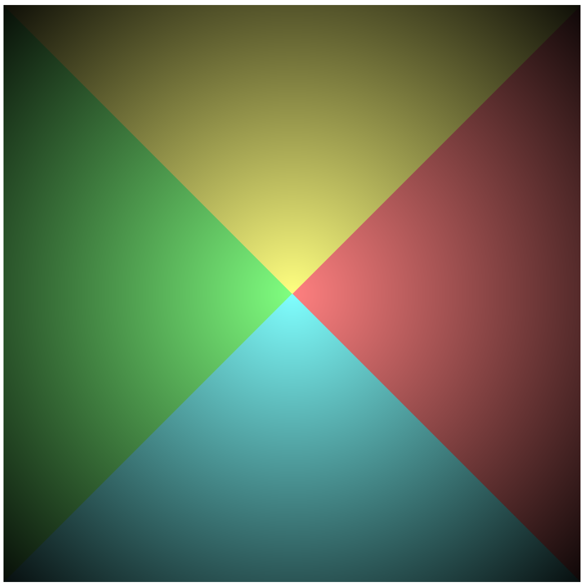
$$f(x)^* = f(\bar{x}) = f(x).$$

However, only points on the unit circle are equal to their own inverse. □

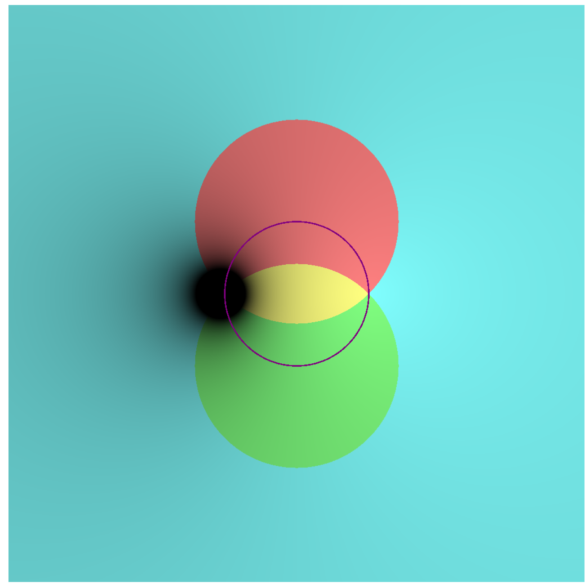
In general such rational functions are not invertible. In order to create a one-to-one correspondence between curves that are symmetric about the real axis and anallagmatic curves we must restrict our attention to linear fractional transformations. Since any linear fractional transformation meeting the hypothesis of Theorem 2.8 will suffice, we select one of the most classically studied such functions, the Cayley transform. The Cayley transform is defined by

$$f(z) = a \left(\frac{z - \alpha}{z - \bar{\alpha}} \right)$$

where $a = -1$ and $\alpha = i$. This complex rational function arises from Theorem 2.8 in the case where $n = 1$. Note that a couple of different versions of the Cayley transform appear in literature, but they all have the property of taking the real axis to the unit circle. Furthermore, the upper half plane maps to the area inside the disk $\{z : \|z\| < 1\}$, and the lower half plane maps to the area outside of the disk, or $\{z : \|z\| > 1\}$. For a graphical depiction of the Cayley transform see Figure 7.



(a) The complex plane.



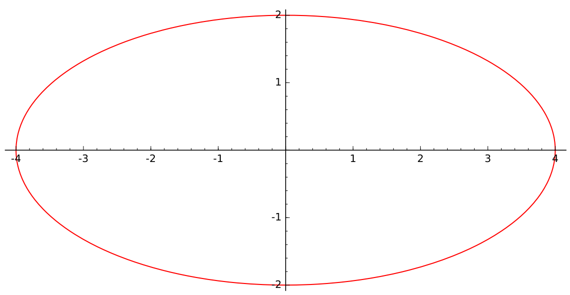
(b) The image of the Cayley transform.

Figure 7: We divide up the plane using this color scheme to show how the Cayley transform maps the complex plane to itself. The dark circular region in Subfigure B comes from points far away from the origin. The purple curve in the left image is the unit circle.

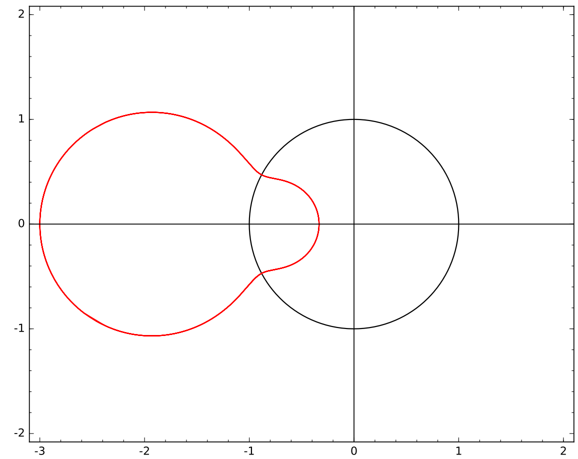
By Theorem 2.8 every curve symmetric about the real axis will map to an anallagmatic curve under the Cayley transform and since this transform is invertible, every such anallagmatic curve arises this way. For an example see Figure 8.

3 Inversion with Respect to the Unit Hyperbola

We now define the concept of inversion about the unit hyperbola centered at the origin. Although inversion can be defined about a large class of curves we choose the unit hyperbola due to the similarities to the unit circle. These similarities will be made clear through the use of the split-complex numbers in Section 3.2.



(a) An ellipse that is symmetric on the real axis.



(b) The image of the ellipse under the Cayley transform.

Figure 8: The ellipse in Subfigure A is symmetric on the real axis, so it is mapped to an anallagmatic curve under the Cayley transform.

3.1 Inversion about the Hyperbola

Definition 3.1. Let P and P' be points that lie on the ray from O through R, where O is the origin and R is a point on the unit hyperbola. Then P and P' are *inverse* to each other if $OP \cdot OP' = OR^2$.

Note that in the case of inversion about the unit circle we could have also used the condition $OP \cdot OP' = OR^2$ where R is the point of intersection of the ray OP with the unit circle. However, on the unit circle we always have $OR = 1$ and therefore this is an analogous definition.

Unlike the case of the unit circle, there are infinitely many points where this inversion is not defined. In particular, this geometric definition only applies if we are between the asymptotes of the hyperbola. Namely it only holds for points with $|y| < |x|$. We will denote this inversion function by $\varphi(P) = P'$.

Proposition 3.2. *If $(x, y) \in \mathbb{R}^2$ satisfy $|y| < |x|$, then*

$$\varphi(x, y) = \left(\frac{x}{x^2 - y^2}, \frac{y}{x^2 - y^2} \right).$$

Proof. Note that

$$\left(\frac{x}{x^2 - y^2}, \frac{y}{x^2 - y^2} \right) = \frac{1}{x^2 - y^2}(x, y).$$

By hypothesis $x^2 - y^2 > 0$, so the product of the magnitude of this point and the magnitude of (x, y) is

$$\frac{\sqrt{x^2 + y^2}}{x^2 - y^2}\sqrt{x^2 + y^2} = \frac{x^2 + y^2}{x^2 - y^2}.$$

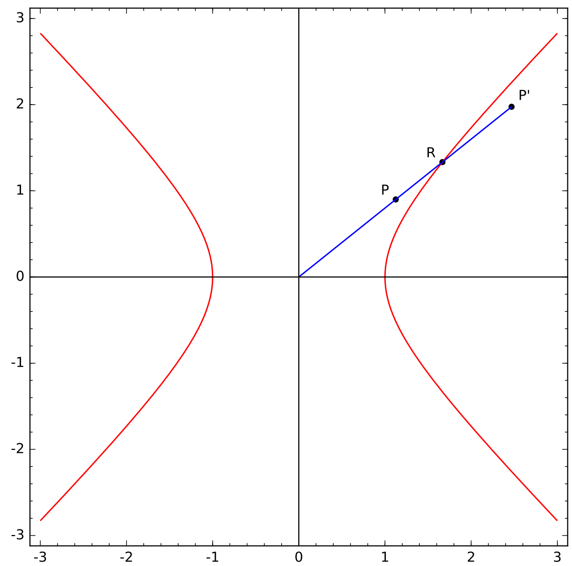


Figure 9: A point P at $\left(\frac{9}{8}, \frac{9}{10}\right)$, and its inverse P' at $\left(\frac{200}{81}, \frac{160}{81}\right)$. The line formed by these two points intersects the hyperbola at the point R $\left(\frac{5}{3}, \frac{4}{3}\right)$.

The line going through the point (x, y) is parameterized by (xt, yt) and intersects the unit hyperbola when $t = \frac{1}{\sqrt{x^2-y^2}}$. In particular, this point of intersection has magnitude

$$\frac{\sqrt{x^2+y^2}}{\sqrt{x^2-y^2}}$$

as desired. $\square$

Although the geometric definition of inverse is technically only defined when $|y| < |x|$, $\varphi(x, y)$ is algebraically defined at all points on the complement of the asymptotes. Therefore, we will generalize inversion to include all points outside of these asymptotes.

Definition 3.3. For any subset $C \subset \mathbb{R}^2 \setminus \{(x, y) : |x| = |y|\}$ of the plane we define the *inverse* of C to be

$$C' = \{\varphi(P) : P \in C\}.$$

As in the circle case we will speak of the inverse of arbitrary subsets of the plane. However, rather than just omitting the origin before taking the inverse we will omit the asymptotes without further mention.

We can examine both parametrically defined and implicitly defined curves with this algebraic definition. For an implicit curve C defined by the equation $f(x, y) = 0$, the inverse curve C' is defined by the equation

$$f\left(\frac{x}{x^2-y^2}, \frac{y}{x^2-y^2}\right) = 0.$$

For a parametric curve C defined by $(x(t), y(t))$, the inverse curve C' is given by the parametric equation

$$\left(\frac{x(t)}{x(t)^2 - y(t)^2}, \frac{y(t)}{x(t)^2 - y(t)^2}\right).$$

For an example of curve and its inverse see Figure 10.

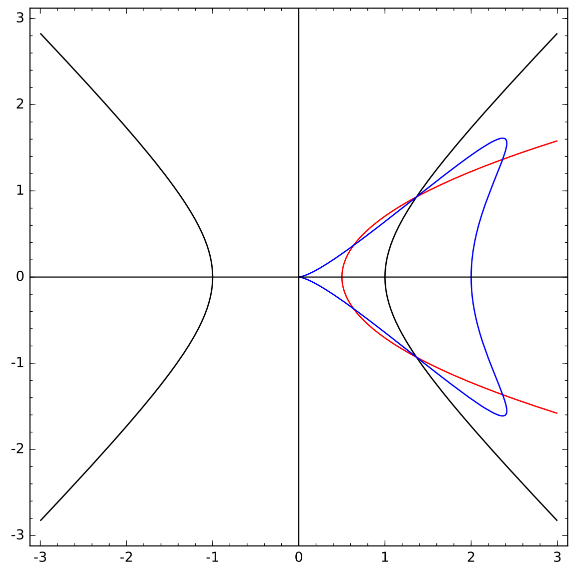


Figure 10: The red parabola is defined by $(t^2 + 1/2, t)$. Its inverse, in blue, is defined by the equation $\left(\frac{4t^2+2}{4t^4+1}, \frac{4t}{4t^4+1}\right)$.

Definition 3.4. Any curve that is its own inverse with respect to the unit hyperbola is *anallagmatic* with respect to the unit hyperbola.

In order to construct some examples of anallagmatic curves we first introduce the split-complex numbers.

3.2 The Split-Complex Numbers

Definition 3.5. The *split-complex numbers* are a ring in which each number is of the form $x+jy$ where $x, y \in \mathbb{R}$ and j is a unit such that $j^2 = 1$. We will denote the set of split-complex numbers as $\mathbb{G}$.

The split-complex numbers share similarities with the complex numbers, the main difference between them being that $j^2 = 1$ in the split-complex numbers rather than $i^2 = -1$ in the complex case. Conjugates of split-complex numbers exist; the conjugate of a split-complex number $w = x + jy$ is $\bar{w} = x - jy$, similar to the conjugate of complex numbers. The modulus of a split complex number w is defined to be $\|w\| = x^2 - y^2$, and the set of split-complex numbers with modulus 1 form the unit hyperbola, comparable to how the set of complex numbers with modulus 1 make up the unit circle. The multiplicative inverse of a split-complex number w is defined as $1/w$, just like in the complex case, and the multiplicative inverse exists so long as w does not have modulus 0. There is also an analogue to Euler's formula for split-complex numbers: $e^{j\theta} = \cosh(\theta) + j\sinh(\theta)$, where cosh is the hyperbolic cosine function and sinh is hyperbolic sine.

For a more extensive discussion of the split-complex numbers see Antonuccio [1], Borota and Osler [3], Segre [8], and Sobczyk [9].

We can now define inversion about the unit hyperbola in terms of the split-complex numbers. This is analogous to the inversion formula over the complex numbers.

Proposition 3.6. *For $w \in \mathbb{G} \setminus \{w : \|w\| = 0\}$, the inverse of w with respect to the unit hyperbola is $1/\bar{w}$.*

Proof. The proof follows immediately from Proposition 3.2. □

We will denote the inverse of a split-complex number w with respect to the unit hyperbola as w^*.

3.3 Rational Function Transforms

Within the complex plane, we have shown that rational functions that satisfy certain properties allow us to construct a relationship between complex conjugates in the plane and points that are inverse with respect to the unit circle.

A similar property can be shown for the split-complex plane as well. By applying certain rational transforms to the split-complex plane, we can map split-complex conjugates to inverse points with respect to the unit hyperbola.

Theorem 3.7. *Let $f(w)$ be a rational function of the form*

$$f(w) = a \prod_{k=1}^{n} \left(\frac{w - \alpha_k}{w - \bar{\alpha}_k} \right),$$

where the $\alpha_k \in \mathbb{G}$ and $a \in \{1, -1\}$. Then for any point $w_0 \in \mathbb{G}$ where $f(w_0)^*$ is defined, $f(w_0)^* = f(\bar{w}_0)$.

Proof. The proof is the same as that of Theorem 2.8. □

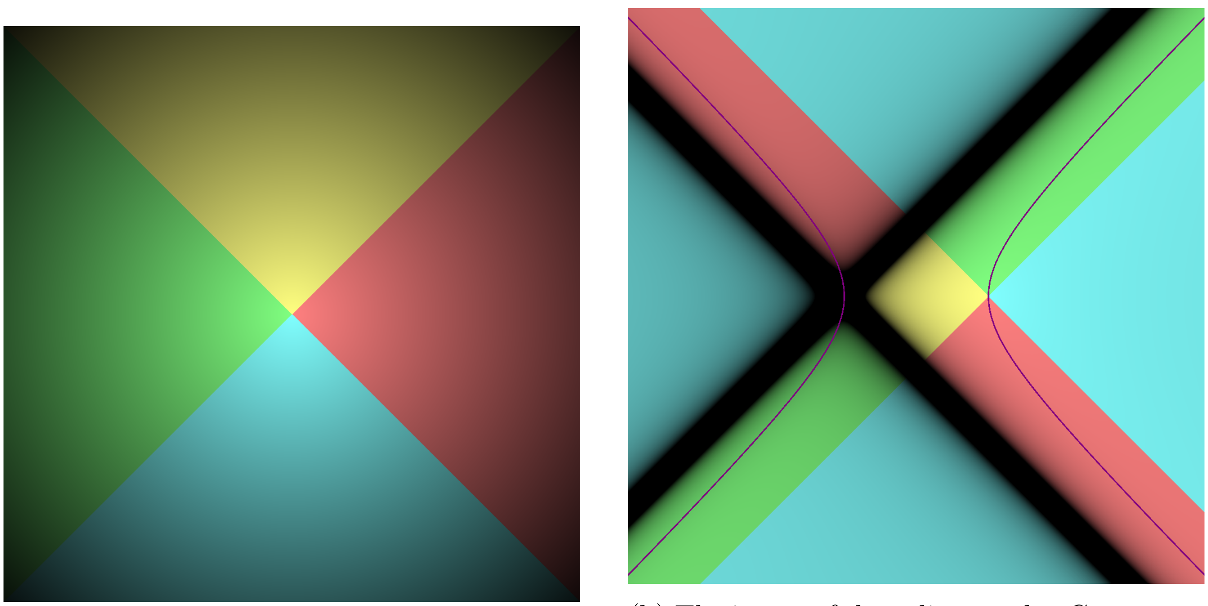
Like the complex case, by using a linear fractional transformation meeting the conditions of Theorem 3.7 we may construct anallagmatic curves about the unit hyperbola. The difference is that curves are restricted to those that are symmetric about a specific part of the real axis, due to the fact that some points in the plane will be mapped to the region $|y| \geq |x|$ which has no geometrically defined inverse. The specific transform we will use is the one most similar to the Cayley transform.

We will refer to the following map

$$f(w) = a\left(\frac{w-\alpha}{w-\bar{\alpha}}\right)$$

with $a = -1$ and $\alpha = j$ as the split-complex Cayley transform. Similar to the Cayley transform, this rational function is derived from Theorem 3.7 in the case that $n = 1$.

Unlike the Cayley transform in the complex plane, the split-complex version does not map the entire plane to the region bounded within the asymptotes of the unit hyperbola. Instead, the transform maps the square with vertices $(1,0),(0,1),(-1,0),(0,-1)$ to the region bounded by the asymptotes of the right half of the unit hyperbola, defined by $\{x^2 - y^2 = 1 | x > 0\}$. See Figure 11 for a graphical representation of this map.

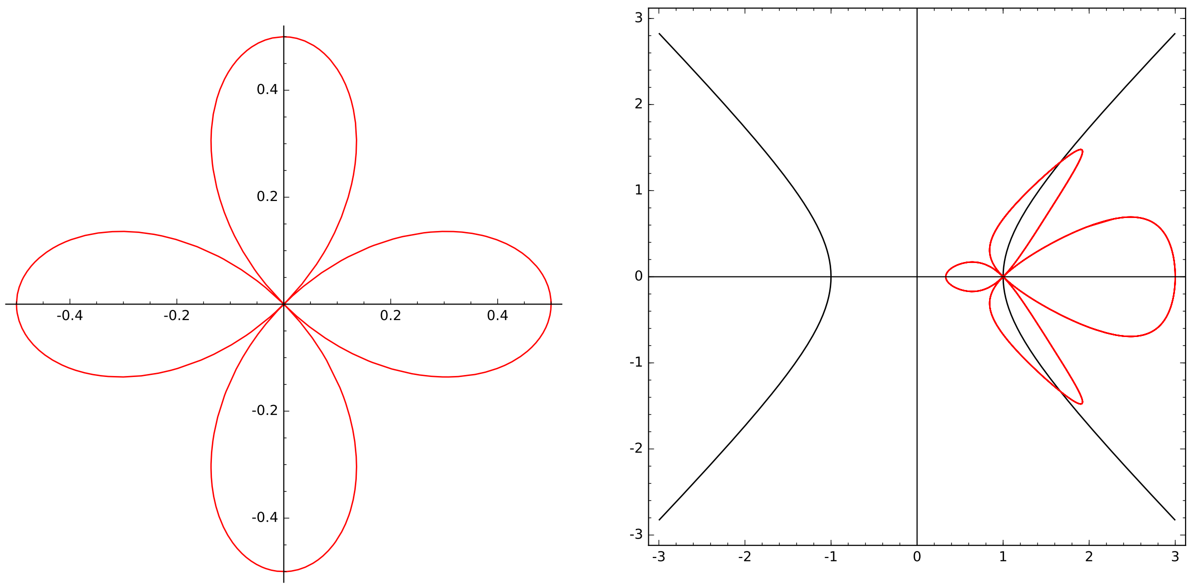


(a) The split-complex plane.

(b) The image of the split-complex Cayley transform.

Figure 11: Similar to before, we use this divided color scheme to show how the split-complex Cayley transform maps the split-complex plane to itself. The purple curve in the right image is the unit hyperbola. The dark regions correspond to the image of points far from the origin in the standard metric.

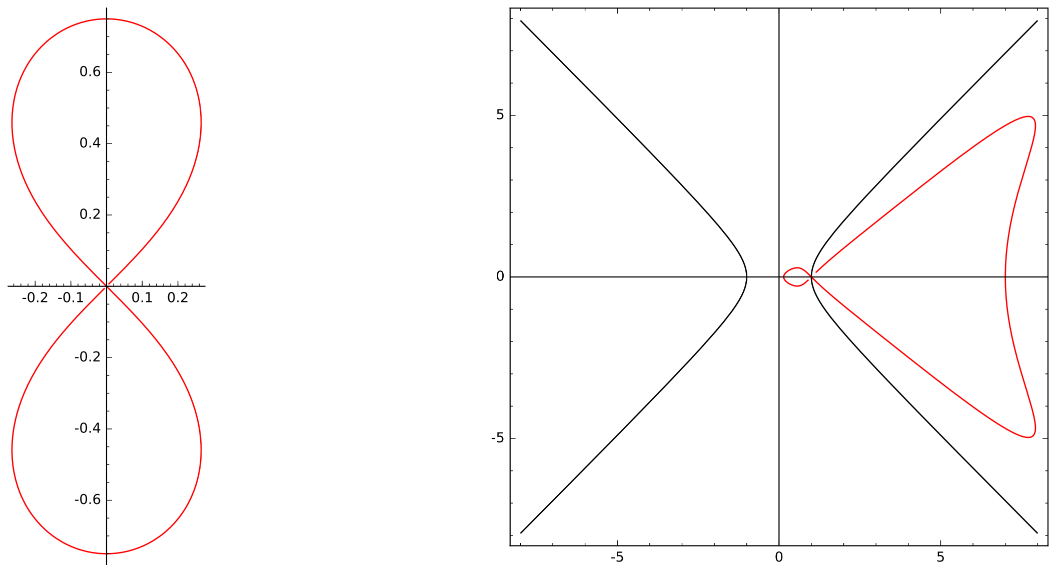
Just as in the case of the Cayley transform over the complex numbers, we can use the split-complex Cayley transform to construct anallagmatic curves by taking the image of any function symmetric about the real axis. See Figures 12 and 13 for examples.



(a) A symmetric curve in the plane defined by the equation $r(\theta) = \frac{1}{2}\cos(2\theta)$.

(b) The image of the curve under the split-complex version of the Cayley transform.

Figure 12: The equation in Subfigure A is expressed in polar form due to its simplicity in depicting the curve. Different scales are used for each figure in order to show the behavior of each curve.



(a) A symmetric curve in the plane defined by the equation $r(\theta) = \frac{3}{4}\sqrt{-\cos(2\theta)}$.

(b) The image of the curve under the split-complex version of the Cayley transform.

Figure 13: For the same reasons stated in Figure 12, the equation in Subfigure A is expressed in polar form, and different scales are in use.

4 Appendix

Figure 4. $$\frac{-(2y^4+4x^2y^2-6y^2+2x^4-x^2)}{2(x^2+y^2)^2}=0$$

Figure 5A. $$\left(-\frac{2\cos(t)+2}{\sin(t)+2\cos(t)+3}, \frac{\cos(t)+2}{\sin(t)+2\cos(t)+3}\right)$$

Figure 5B. $$\left(-\frac{4\cos(2t)+1}{4\cos(2t)-4\sin(t)\sqrt{-\cos(2t)}-1}, -\frac{4\cos(t)\sqrt{-\cos(2t)}}{4\cos(2t)-4\sin(t)\sqrt{-\cos(2t)}-1}\right)$$

Figure 12B. $$\left(-\frac{\cos(6t)+3\cos(2t)+16}{\cos(6t)-8\sin(3t)+3\cos(2t)+8\sin(t)-16}, \frac{8\cos(3t)+8\cos(t)}{\cos(6t)-8\sin(3t)+3\cos(2t)+8\sin(t)-16}\right)$$

Figure 13B. $$\left(-\frac{9\cos(4t)-23}{9\cos(4t)+48\sin(t)\sqrt{-\cos(2t)}+41}, -\frac{48\cos(t)\sqrt{-\cos(2t)}}{9\cos(4t)+48\sin(t)\sqrt{-\cos(2t)}+41}\right)$$

References

[1] F. Antonuccio. Semi-complex analysis and mathematical physics. *arXiv preprint gr-qc/9311032*, 1993.

[2] D. E. Blair. *Inversion Theory and Conformal Mapping*, volume 9. American Mathematical Soc., 2000.

[3] N. A. Borota and T. J. Osler. Functions of a spacetime variable. *Mathematics and Computer Education*, 36(3):231–239, 2002.

[4] N. A. Childress. Inversion with respect to the central conics. *Mathematics Magazine*, 38(3):147–149, 1965.

[5] J. L. Coolidge. *A Treatise on the Circle and the Sphere*. American Mathematical Soc., 2004.

[6] T. Needham. *Visual complex analysis*. Oxford University Press, 1998.

[7] J. L. Ramírez. An introduction to inversion in an ellipse. *arXiv preprint arXiv:1309.6378*, 2013.

[8] A. K. G. Segre. An introduction to hyperbolic analysis. *arXiv preprint math-ph/0507053*, 2005.

[9] G. Sobczyk. The hyperbolic number plane. *The College Mathematics Journal*, 26(4):268–280, 1995.