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PERFECT ZERO-DIVISOR GRAPHS OF $\mathbb{Z}_n$

Bennett Smith

Abstract. The zero-divisor graph of a ring R, denoted $\Gamma(R)$, is the graph whose vertex set is the collection of zero-divisors in R, with edges between two distinct vertices u and v if and only if $uv = 0$. In this paper, we restrict our attention to $\Gamma(\mathbb{Z}_n)$, the zero-divisor graph of the ring of integers modulo n. Specifically, we determine all values of n for which $\Gamma(\mathbb{Z}_n)$ is perfect. Our classification depends on the prime factorization of n, with relatively simple prime factorizations corresponding to perfect graphs. In fact, for values of n with at most two distinct prime factors, $\Gamma(\mathbb{Z}_n)$ is perfect; for n with at least 5 distinct prime factors, $\Gamma(\mathbb{Z}_n)$ is not perfect; and for n with either three or four distinct prime factors, $\Gamma(\mathbb{Z}_n)$ is perfect only in the cases where $n = p^a qr$ and $n = pqrs$ for distinct primes p, q, r, s and positive integer a. In proving our results, we make heavy use of the Strong Perfect Graph Theorem.

1 Introduction

Consider the set of zero-divisors of $\mathbb{Z}_n$, denoted $\mathrm{ZD}(\mathbb{Z}_n)$. The zero-divisor graph of $\mathbb{Z}_n$, denoted $\Gamma(\mathbb{Z}_n)$, is the graph whose vertex set is $\mathrm{ZD}(\mathbb{Z}_n)$, where distinct vertices u and v are adjacent if and only if $uv \equiv_n 0$. For example, Figure 1.1 illustrates the zero-divisor graph of $\mathbb{Z}_{12}$. Zero-divisor graphs can be defined more generally for any ring R, but the focus of this paper is on zero-divisor graphs of $\mathbb{Z}_n$.

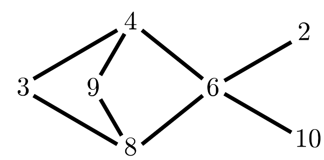

Figure 1.1: The zero-divisor graph of $\mathbb{Z}_{12}$.

Zero-divisor graphs were first introduced in 1988 by Beck, where the primary focus was on the coloring of the graphs [4]. Several decades later, the concept was further explored by Anderson and Livingston, whose work connecting the areas of commutative ring theory and graph theory continues to encourage further research on zero-divisor graphs [2]. Beck's focus is applicable to more recent research done on proper colorings of zero-divisor graphs, such as by Duane [6]. The zero-divisor graph survey paper by Anderson, Axtell, and Stickles summarizes much of this research [1].

Endean, Henry, and Manlove attempted to determine all values of n for which the zero-divisor graph of $\mathbb{Z}_n$ is perfect [7]. Although they deserve much credit for introducing this question, their work contains mistaken results. Here, we correctly determine all values of n for which $\Gamma(\mathbb{Z}_n)$ is perfect. To do so, we make use of the Strong Perfect Graph Theorem, a beautifully simple characterization of perfect graphs, proved by Chudnovsky, Robertson, Seymore, and Thomas [5]. We also use a simplification of the zero-divisor graph of $\mathbb{Z}_n$, which we refer to as the **zero-divisor type graph** of $\mathbb{Z}_n$. Recently, we have learned that this concept has been discovered before and used in earlier works. For example, these simplified graphs are presented more generally and referred to as **compressed graphs** by Axtell, Baeth, and Stickles [3] and by Weber [9]. Additionally, these simplified graphs are referred to as **graphs of equivalency classes of zero-divisors** in $\mathbb{Z}_n$ by Spiroff and Wickham [8]. As others before us, we have found this simplified view of zero-divisor graphs to be very useful.

In Section 2, we discuss the definition of a perfect graph. In Section 3, we discuss the simplification of zero-divisor graphs to zero-divisor type graphs. Then in Section 4, we prove that the zero-divisor graph is perfect if and only if the zero-divisor type graph is perfect. Using this theorem and the Strong Perfect Graph Theorem, in Sections 5 and 6 we completely determine all values of n for which the zero-divisor graph of $\mathbb{Z}_n$ is perfect, leading to our main theorem.

Main Theorem. *The zero-divisor graph of $\mathbb{Z}_n$, $\Gamma(\mathbb{Z}_n)$, is perfect if and only if n has one of the following four forms.*

1. $n = p^a$ *for a prime* p *and positive integer* a
2. $n = p^a q^b$ *for distinct primes* p, q *and positive integers* a, b
3. $n = p^a qr$ *for distinct primes* p, q, r *and positive integer* a
4. $n = pqrs$ *for distinct primes* p, q, r, s.

2 Perfectness of a Graph

In this section, we define what it means for a graph to be perfect and introduce a simple way to determine the perfectness of any graph. The definition of a perfect graph relies upon the following two terms. The **chromatic number** of a graph G is the minimum number of colors required to color the vertices of G such that no two adjacent vertices have the same color. The **clique number** of graph G is the size of the largest complete subgraph of G. This leads to the following definition.

Definition. A **perfect graph** is a graph G for which every induced subgraph of G has chromatic number equal to its clique number.

Example 2.1. Consider the graph of a 5-cycle, shown in Figure 2.1. The chromatic number of the graph is 3, while the clique number is 2. Thus, the graph of a 5-cycle is not perfect.

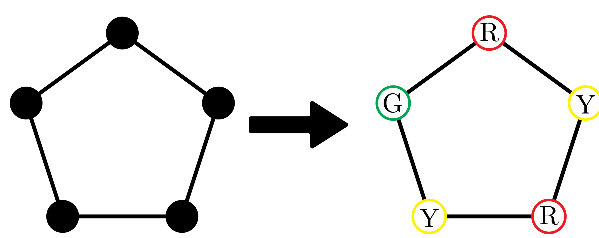

Figure 2.1: The most efficient coloring of a 5-cycle. The number of colors required is three, so the chromatic number is 3.

Now, consider the graph of a 4-cycle, shown in Figure 2.2. Here, the chromatic number and the clique number of the graph are 2. Thus, the chromatic number and the clique number are equal. For the graph to be perfect the clique number and chromatic number must be equal for every induced subgraph as well. This is easily verified, and hence the graph of a 4-cycle is perfect.

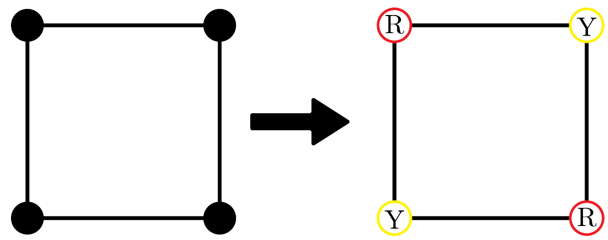

Figure 2.2: The most efficient coloring of a 4-cycle. The number of colors required is two, so the chromatic number is 2.

Another way to check the perfectness of these two graphs is with the Strong Perfect Graph Theorem, which reaffirms that *no* cycle of odd length at least 5 is perfect, while *every* cycle of even length at least 4 is perfect [5]. First, note that a **hole** of a graph G is an induced subgraph in the form of a cycle of length at least 4; whereas an **antihole** is an induced subgraph of G whose complement is a hole in $\overline{G}$. Also, note that the length of a hole or antihole refers to the number of vertices it contains.

Strong Perfect Graph Theorem. *A graph G is perfect if and only if it contains no hole or antihole of odd length.*

3 Zero-Divisor Type Graphs

In this section, we define a graph called the **zero-divisor type graph** of $\mathbb{Z}_n$, denoted $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. We note that the zero-divisor type graph is identical to the **graph of equivalency classes of zero-divisors** in $\mathbb{Z}_n$. Graphs of equivalency classes of zero-divisors in a ring R were first defined by Spiroff and Wickham [8]. Other authors have referred to these graphs as **compressed zero-divisor graphs** [3, 9]. Here, we discuss the same concept in the case were $R = \mathbb{Z}_n$ and we refer to these graphs as zero-divisor type graphs.

We will see that the zero-divisor type graph is very closely related to the zero-divisor graph. However, the zero-divisor type graph is much simper in most cases, while maintaining the property of perfectness (proven in Section 4). We begin by defining the vertex and edge sets of zero-divisor type graphs. Note that a nontrivial divisor of n is a divisor of n that is neither 1 nor n.

Vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. The set of vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is given by

$$\{T_d : d \text{ is a nontrivial divisor of } n\},$$

where T_d is the subset of $\mathrm{ZD}(\mathbb{Z}_n)$ defined by

$$T_d = \{k \in \mathrm{ZD}(\mathbb{Z}_n) : \gcd(k, n) = d\}.$$

Edges of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. Two distinct vertices T_{d_1} and T_{d_2} are connected by an edge in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ if and only if $d_1 d_2 \equiv_n 0$.

We refer to each vertex of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ as a *zero-divisor type*, or simply a *type*. We say that a zero-divisor k is of type d or in type T_d to mean that $k \in T_d$. We use the terms zero-divisor type graph and type graph to refer to the same thing. Before making a few key observations about the type graph of $\mathbb{Z}_n$, we present an example.

Example 3.1. Consider $\mathbb{Z}_{12}$. The set of zero-divisors in $\mathbb{Z}_{12}$ is

$$\mathrm{ZD}(\mathbb{Z}_{12}) = \{2, 3, 4, 6, 8, 9, 10\},$$

and the zero-divisor graph, $\Gamma(\mathbb{Z}_{12})$, can be seen in Figure 3.1.

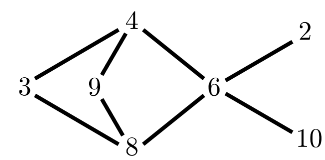

Figure 3.1: The zero-divisor graph of $\mathbb{Z}_{12}$.

Note that the nontrivial divisors of 12 are 2, 3, 4, and 6. Hence, the set of vertices of the type graph of $\mathbb{Z}_{12}$ is

$$\{T_2, T_3, T_4, T_6\},$$

where each type is as given below:

$$\begin{aligned} T_2 &= \{k \in \mathbb{Z}_{12} : \gcd(k, 12) = 2\} = \{2, 10\}, \\ T_3 &= \{k \in \mathbb{Z}_{12} : \gcd(k, 12) = 3\} = \{3, 9\}, \\ T_4 &= \{k \in \mathbb{Z}_{12} : \gcd(k, 12) = 4\} = \{4, 8\}, \text{ and} \\ T_6 &= \{k \in \mathbb{Z}_{12} : \gcd(k, 12) = 6\} = \{6\}. \end{aligned}$$

There are exactly three edges in $\Gamma^{\mathrm{T}}(\mathbb{Z}_{12})$:

$$\begin{aligned} &T_2 \sim T_6 \text{ since } 2 \cdot 6 \equiv_{12} 0, \\ &T_4 \sim T_6 \text{ since } 4 \cdot 6 \equiv_{12} 0, \text{ and} \\ &T_3 \sim T_4 \text{ since } 3 \cdot 4 \equiv_{12} 0. \end{aligned}$$

Thus, we observe the type graph of $\Gamma^{\mathrm{T}}(\mathbb{Z}_{12})$ shown in Figure 3.2.

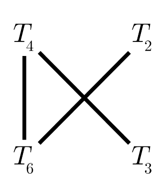

Figure 3.2: The zero-divisor type graph of $\mathbb{Z}_{12}$.

Note that $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is simpler than $\Gamma(\mathbb{Z}_n)$ while retaining much of the same information. In the case of $n = 12$, the change is not very significant, but the simplification becomes much greater when we compare $\Gamma(\mathbb{Z}_n)$ to $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ for larger values of n and for values of n with more complicated prime factorizations. For example, $\Gamma(\mathbb{Z}_{2738})$ has 1405 vertices (one for every zero-divisor in $\mathbb{Z}_{2738}$), while $\Gamma^{\mathrm{T}}(\mathbb{Z}_{2738})$ has only 4 vertices (one for every nontrivial divisor of 2738). Now, we make a few useful observations about the type graph of $\mathbb{Z}_n$.

Observation 3.1. The vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ partition the zero-divisors of $\mathbb{Z}_n$ into types. That is, every zero-divisor is in one, and only one, type.

Observation 3.2. Given types T_{d_1} and T_{d_2}, we have that $T_{d_1} \sim T_{d_2}$ in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ if and only if $v_1 \sim v_2$ in $\Gamma(\mathbb{Z}_n)$ for every pair of vertices v_1 and v_2 with $v_1 \in T_{d_1}$ and $v_2 \in T_{d_2}$.

Observation 3.3. Given types T_{d_1} and T_{d_2}, we have that $T_{d_1} \nsim T_{d_2}$ in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ if and only if $v_1 \nsim v_2$ in $\Gamma(\mathbb{Z}_n)$ for every pair of vertices v_1 and v_2 with $v_1 \in T_{d_1}$ and $v_2 \in T_{d_2}$.

Observation 3.4. If vertices u and v of $\Gamma(\mathbb{Z}_n)$ are in the same type T_d, then for any other vertex w of $\Gamma(\mathbb{Z}_n)$, $u \sim w$ in $\Gamma(\mathbb{Z}_n)$ if and only if $v \sim w$ in $\Gamma(\mathbb{Z}_n)$.

Observation 3.1 states precisely that we can define an equivalence relation $*$ on $\mathrm{ZD}(\mathbb{Z}_n)$ such that for any $a, b \in \mathrm{ZD}(\mathbb{Z}_n)$, $a * b$ if and only if a and b are in the same type. Furthermore, $*$ behaves well with the operation described by Observations 3.2–3.4. This equivalence relation is similar to the one used to determine the classes of zero-divisors in the graph of equivalence classes of zero-divisors [8]. Now, we present the next example to introduce an alternate way to label the zero-divisor types of $\mathbb{Z}_n$ (the vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$) using tuples.

Example 3.2. Consider $\mathbb{Z}_n$ where $n = p^2qr$ for distinct primes p, q, r. The number of zero-divisors of $\mathbb{Z}_n$ depends on the specific prime factors. However, regardless of the specific values of these primes, the set of nontrivial divisors of n is $\{p, q, r, p^2, pq, pr, qr, p^2q, p^2r, pqr\}$. Thus, the set of vertices (or types) of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is

$$\{T_p, T_q, T_r, T_{p^2}, T_{pq}, T_{pr}, T_{qr}, T_{p^2q}, T_{p^2r}, T_{pqr}\}.$$

We simplify the labels in the above set by using triples, writing

$$T_d = (a, b, c), \text{ where } d = p^a q^b r^c.$$

Using this convention, the set of vertices (or types) of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ can now be written as

$$\begin{gathered}\{(1,0,0),(0,1,0),(0,0,1),(2,0,0),(1,1,0),\\(1,0,1),(0,1,1),(2,1,0),(2,0,1),(1,1,1)\}.\end{gathered}$$

Note that two vertices (a, b, c) and (x, y, z) are adjacent in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ if and only if

$$a + x \geq 2,\ b + y \geq 1,\ \text{and } c + z \geq 1.$$

That is, $(a, b, c) \sim (x, y, z)$ in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ if and only if respective components sum to at least the full power of the associated prime in the prime factorization of $n = p^2qr$. The type graph of $\mathbb{Z}_{p^2qr}$ is shown in Figure 3.3.

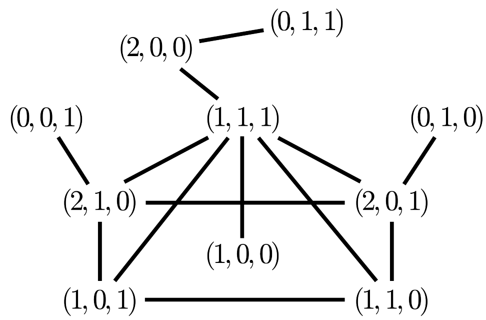

Figure 3.3: The zero-divisor type graph of $\mathbb{Z}_{p^2qr}$.

4 Perfectly Related

In this section, we will use the Strong Perfect Graph Theorem to prove that the zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ is perfect if and only if the zero-divisor type graph of $\Gamma(\mathbb{Z}_n)$ is perfect. The theorem (and supporting lemmas) extend to the zero-divisor graph and the zero-divisor type graph of any commutative ring, given that the zero-divisor types for the ring are defined in such a way that Observations 3.1–3.4 still hold true. Recall that the length of a hole or antihole refers to the number of vertices that it contains.

Lemma 4.1. *In $\Gamma(\mathbb{Z}_n)$, there cannot exist a hole or antihole of length greater than four that has two or more vertices from the same type.*

Proof. Assume that we have a hole or antihole H of length greater than four. Let at least two of the vertices, say u and v, be in the same type, T_d. Since there are at least five vertices contained in H, there must be at least one other vertex that is adjacent to u, but nonadjacent to v. That is, there exists a vertex w of H such that such that $u \sim w$ and $v \nsim w$. However, since $u, v \in T_d$, by Observation 3.4 we have that $v \sim w$, a contradiction. Thus, there cannot exist a hole or antihole of length greater than four that has two or more vertices from the same type. $\square$

Lemma 4.2. *A hole or antihole of length $k > 4$ exists in $\Gamma(\mathbb{Z}_n)$ if and only if a hole or antihole of length k exists in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.*

Proof. Assume $\Gamma(\mathbb{Z}_n)$ contains a hole H of length $k > 4$. Let the set of vertices of H be $\{v_1, v_2, \ldots, v_k\}$ where

$$v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{k-1} \sim v_k \sim v_1$$

are the only adjacencies of H. For each $i \in \{1, 2, \ldots, k\}$, let T_{d_i} be the type where $v_i \in T_{d_i}$. Then, Lemma 4.1 implies $T_{d_i} \neq T_{d_j}$ when $i \neq j$. Hence, we have k distinct types. Now, by

Observations 3.2 and 3.3,

$$T_{d_1} \sim T_{d_2} \sim T_{d_3} \sim \ldots \sim T_{d_{k-1}} \sim T_{d_k} \sim T_{d_1}$$

are the only adjacencies among $\{T_{d_1}, T_{d_2}, \ldots, T_{d_k}\}$. Thus, $\{T_{d_1}, T_{d_2}, \ldots, T_{d_k}\}$ forms a hole of length k in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. Similarly, if $\Gamma(\mathbb{Z}_n)$ contains an antihole of length $k > 4$, we can see that this proof would be the same except adjacencies would be replaced by nonadjacencies.

Now, assume $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains a hole H of length $k > 4$. Let the set of vertices of H be $\{T_{d_1}, T_{d_2}, \ldots, T_{d_k}\}$ where

$$T_{d_1} \sim T_{d_2} \sim T_{d_3} \sim \ldots \sim T_{d_{k-1}} \sim T_{d_k} \sim T_{d_1}$$

are the only adjacencies of H. For each $i \in \{1, 2, \ldots, k\}$, select a zero-divisor v_i with $v_i \in T_{d_i}$. Now, by Observations 3.2 and 3.3,

$$v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{k-1} \sim v_k \sim v_1$$

are the only adjacencies among $\{v_1, v_2, \ldots, v_k\}$. Thus, $\{v_1, v_2, \ldots, v_k\}$ forms a hole of length k in $\Gamma(\mathbb{Z}_n)$. Again, if $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains an antihole of length $k > 4$, we can see that this proof would be the same except adjacencies would be replaced by nonadjacencies.

Thus, a hole or antihole of length $k > 4$ exists in $\Gamma(\mathbb{Z}_n)$ if and only if a hole or antihole of length k exists in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. □

Now, we have everything we need to prove that the zero-divisor graph is perfect if and only if the zero-divisor type graph is perfect.

Theorem 4.1. $\Gamma(\mathbb{Z}_n)$ *is perfect if and only if* $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ *is perfect.*

Proof. By Lemma 4.2, $\Gamma(\mathbb{Z}_n)$ contains a hole or antihole of odd length if and only if $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ does. Hence, by the Strong Perfect Graph Theorem, $\Gamma(\mathbb{Z}_n)$ is perfect if and only if $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is perfect. □

In the next two sections, we will use Theorem 4.1 and the Strong Perfect Graph Theorem extensively as we work to completely determine the values of n for which $\Gamma(\mathbb{Z}_n)$ is perfect.

5 Values of n for which $\Gamma(\mathbb{Z}_n)$ is Perfect

In this section, we will show that for values of n with relatively simple prime factorizations, $\Gamma(\mathbb{Z}_n)$ is perfect. Specifically, if n has a prime factorization in one of the following forms, then $\Gamma(\mathbb{Z}_n)$ is a perfect graph:

$$\begin{aligned}
&n = p^a \text{ for a prime } p \text{ and positive integer } a,\\
&n = p^a q^b \text{ for distinct primes } p, q \text{ and positive integers } a, b,\\
&n = p^a qr \text{ for distinct primes } p, q, r \text{ and positive integer } a, \text{ and}\\
&n = pqrs \text{ for distinct primes } p, q, r, s.
\end{aligned}$$

In the next section, we will show that, in fact, these are the *only* values of n for which $\Gamma(\mathbb{Z}_n)$ is perfect. Before we begin proving each of these cases, we make some observations about holes and antiholes that will be helpful.

Observation 5.1. Given any two vertices v_1 and v_2 of a hole or antihole H of length greater than 4, there exists at least one other vertex u of H such that $u \sim v_1$ and $u \nsim v_2$.

Observation 5.2. No hole can contain a cycle of length 3.

Observation 5.3. No antihole can contain three vertices that are pairwise nonadjacent.

Observation 5.4. In any hole or antihole of length greater than 4, every vertex has two non-neighbors that are neighbors of each other.

Observation 5.5. Given any vertex v of a hole or antihole H of length greater than 4, we can find a path of length 5 contained in H and centered at v so that

$$a \sim b \sim v \sim c \sim d$$

with

$$a \nsim c, \; b \nsim c, \; \text{and } b \nsim d.$$

We will use these observations in the following subsections to prove that $\Gamma(\mathbb{Z}_n)$ is indeed perfect for the values of n listed above.

5.1 $\Gamma(\mathbb{Z}_n)$ for $n = p^a$

Throughout this subsection, we assume $n = p^a$ for a prime p and positive integer a. We work with the type graph, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, as discussed in Section 3. Recall that the collection of vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is

$$\{T_d : d \text{ is a nontrivial divisor of } n\}.$$

Note that each nontrivial divisor of n is of the form $d = p^x$. For simplicity, we denote T_d with the singleton (x), where $d = p^x$. Using this notation, we have

$$(x_1) \sim (x_2) \text{ in } \Gamma^{\mathrm{T}}(\mathbb{Z}_n)$$

if and only if

$$x_1 + x_2 \geq a.$$

To prove that $\Gamma(\mathbb{Z}_n)$ is perfect, we begin by proving the following lemma.

Lemma 5.1. *If $n = p^a$ for a prime p and positive integer a, then there exists no hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.*

Proof. Assume H is either a hole or an antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. Then, by Observation 5.5, we can choose a path of length 5 contained in H as follows:

$$(x_1) \sim (x_2) \sim (x_3) \sim (x_4) \sim (x_5)$$

with

$$(x_1) \nsim (x_4) \text{ and } (x_2) \nsim (x_5).$$

It then follows that

$$x_1 + x_4 < a \leq x_1 + x_2 \text{ since } (x_1) \nsim (x_4) \text{ and } (x_1) \sim (x_2).$$

Thus,

$$x_4 < x_2.$$

On the other hand, we have

$$x_2 + x_5 < a \leq x_4 + x_5 \text{ since } (x_2) \nsim (x_5) \text{ and } (x_4) \sim (x_5).$$

Thus,

$$x_2 < x_4.$$

So, we have $x_4 < x_2$ and $x_2 < x_4$, which is clearly a contradiction. Therefore, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of length greater than 4. $\square$

In fact, using an approach similar to the one taken in the proof of Lemma 5.1, it can be shown that no hole can exist in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ whatsoever. Now, using Theorem 4.1 and the Strong Perfect Graph Theorem, we have everything needed to prove that $\Gamma(\mathbb{Z}_n)$ is perfect.

Theorem 5.1. *If $n = p^a$ for a prime p and positive integer a, then $\Gamma(\mathbb{Z}_n)$ is perfect.*

Proof. By Lemma 5.1, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of length greater than 4. This is equivalent to saying $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of odd length. Therefore, by the Strong Perfect Graph Theorem, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is perfect. Thus, by Theorem 4.1, $\Gamma(\mathbb{Z}_n)$ is perfect. $\square$

5.2 $\Gamma(\mathbb{Z}_n)$ for $n = p^a q^b$

Throughout this subsection, we assume $n = p^a q^b$ for distinct primes p, q and positive integers a, b. We work with the type graph, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, whose vertices we denote with ordered pairs, as discussed in Section 3. Recall that the collection of vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is

$$\{T_d : d \text{ is a nontrivial divisor of } n\}.$$

Note that each nontrivial divisor of n is of the form $d = p^x q^y$. For simplicity, we denote T_d with the ordered pair (x, y), where $d = p^x q^y$. Using this notation, we have

$$(x_1, y_1) \sim (x_2, y_2) \text{ in } \Gamma^{\mathrm{T}}(\mathbb{Z}_n)$$

if and only if

$$x_1 + x_2 \geq a \text{ and } y_1 + y_2 \geq b.$$

Before we prove that the $\Gamma(\mathbb{Z}_n)$ is perfect, we prove a series of four lemmas. The first two lemmas are quite technical.

Lemma 5.2. *With $n = p^a q^b$ for distinct primes p, q and positive integers a, b, if H is a hole or antihole in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, then for any two vertices (x_1, y_1) and (x_2, y_2) of H either we have both $x_1 < x_2$ and $y_1 > y_2$, or we have both $x_1 > x_2$ and $y_1 < y_2$.*

Proof. Assume (x_1, y_1) and (x_2, y_2) are two vertices of H. Then, of course, either $x_1 \leq x_2$ or $x_1 \geq x_2$. Assume first that $x_1 \leq x_2$. Using Observation 5.1, choose a vertex (u, v) of H such that $(u, v) \sim (x_1, y_1)$ but $(u, v) \nsim (x_2, y_2)$. Then $x_2 + u \geq x_1 + u \geq a$ since $x_2 \geq x_1$ and $(x_1, y_1) \sim (u, v)$. Now, since $x_2 + u \geq a$ and $(x_2, y_2) \nsim (u, v)$, we must have $y_2 + v < b$. Thus, $y_2 + v < b \leq y_1 + v$ since $(x_1, y_1) \sim (u, v)$, so $y_2 < y_1$. Thus, if $x_1 \leq x_2$, then $y_1 > y_2$. Similarly, if $x_1 \geq x_2$, then $y_1 < y_2$.

Now, note that if $x_1 = x_2$, then we have both $y_1 < y_2$ and $y_1 > y_2$, which is of course impossible. Therefore, either we have both $x_1 < x_2$ and $y_1 > y_2$, or we have both $x_1 > x_2$ and $y_1 < y_2$. □

Lemma 5.3. *With $n = p^a q^b$ for distinct primes p, q and positive integers a, b, if*

$$(u, v),\ (x_1, y_1),\ \textit{and}\ (x_2, y_2)$$

are three vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ where

$$(u, v) \sim (x_1, y_1),\ (u, v) \sim (x_2, y_2),\ \textit{and}\ (x_1, y_1) \nsim (x_2, y_2),$$

then we have either both $u < x_1$ and $u < x_2$, or both $u > x_1$ and $u > x_2$.

Proof. Assume that (u, v), (x_1, y_1), and (x_2, y_2) are three vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ where

$$(u, v) \sim (x_1, y_1),\ (u, v) \sim (x_2, y_2),\ \text{and}\ (x_1, y_1) \nsim (x_2, y_2).$$

Note that from Lemma 5.2 we know that $u \neq x_1$ and $u \neq x_2$. Suppose that we have neither both $u < x_1$ and $u < x_2$, nor both $u > x_1$ and $u > x_2$. Without loss of generality, assume $u > x_1$ and $u < x_2$. By Lemma 5.2, we have

$$u > x_1 \text{ and } v < y_1,$$

and

$$u < x_2 \text{ and } v > y_2.$$

It follows that

$$x_1 + x_2 > x_1 + u \geq a \text{ since } (u, v) \sim (x_1, y_1),$$

and

$$y_1 + y_2 > v + y_2 \geq b \text{ since } (u, v) \sim (x_2, y_2).$$

Thus $(x_1, y_1) \sim (x_2, y_2)$, which is a contradiction.

Thus, we have either both $u < x_1$ and $u < x_2$, or both $u > x_1$ and $u > x_2$. □

Lemma 5.4. *If $n = p^a q^b$ for distinct primes p, q and positive integers a, b, then there exists no hole of odd length in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.*

Proof. Assume $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ does contain a hole H of odd length. Define subsets of the collection of vertices of H as follows:

$$R = \{(u, v) : u < x \text{ for every neighbor } (x, y) \text{ of } (u, v) \text{ in } H\}, \text{ and}$$
$$B = \{(u, v) : u > x \text{ for every neighbor } (x, y) \text{ of } (u, v) \text{ in } H\}.$$

We claim that every vertex of H is in either R or B. If not, then choose vertex (u, v) of H such that $(u, v) \notin R \cup B$. Since $(u, v) \notin R$, we can choose vertex (x_1, y_1) of H such that $(u, v) \sim (x_1, y_1)$ and $u \geq x_1$. Also, since $(u, v) \notin B$, we can choose vertex (x_2, y_2) of H such that $(u, v) \sim (x_2, y_2)$ and $u \leq x_2$. Unless $(x_1, y_1) \sim (x_2, y_2)$, this contradicts Lemma 5.3. However, if $(x_1, y_1) \sim (x_2, y_2)$, then the vertices (u, v), (x_1, y_1), and (x_2, y_2) form a cycle of length 3 in the hole H, which is not possible as noted in Observation 5.2 above. Thus, every vertex of H is in either R or B.

Now, note that by the definition of R and B, if vertex (u, v) of H is in R, then its neighbors must be in B. So, as we traverse around the vertices of H, the vertices alternate between those contained in R and those contained in B. This is not possible since the number of vertices of H is odd. Thus, we have a contradiction, so $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole of odd length. $\square$

The proof of our next lemma has similarities to the one just presented, but significant differences are necessary so we must present it as a separate proof.

Lemma 5.5. *If $n = p^a q^b$ for distinct primes p, q and positive integers a, b, then there exists no antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.*

Proof. Assume $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains an antihole A of length greater than 4. Define subsets of the collection of vertices of A as follows:

$$R = \{(u, v) : u < x \text{ for every neighbor } (x, y) \text{ of } (u, v) \text{ in } A\}, \text{ and}$$
$$B = \{(u, v) : u > x \text{ for every neighbor } (x, y) \text{ of } (u, v) \text{ in } A\}.$$

We claim that every vertex of A is in either R or B. To see this, take an arbitrary vertex (u, v) of A. Since A is an antihole, we can list *all* the vertices of A which are neighbors of (u, v) as

$$(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k),$$

where

$$(x_1, y_1) \nsim (x_2, y_2) \nsim \ldots \nsim (x_k, y_k).$$

Now, using Lemma 5.3, we have one of the following chains of implications:

$$u < x_1 \Rightarrow u < x_2 \Rightarrow \ldots \Rightarrow u < x_k$$

or

$$u > x_1 \Rightarrow u > x_2 \Rightarrow \ldots \Rightarrow u > x_k.$$

Thus, (u, v) is in either R or B. Also, by the definition of R and B, if two vertices of A are adjacent, then one of the vertices must be in R while the other must be in B. That is, vertices in R are pairwise nonadjacent and vertices in B are pairwise nonadjacent.

Now, since A must have at least 5 vertices, each of which is in either R or B, either A has at least 3 vertices in R or A has at least 3 vertices in B. Therefore, since vertices in R are pairwise nonadjacent and vertices in B are pairwise nonadjacent, we must have three vertices of A that are pairwise nonadjacent. This contradicts Observation 5.3. Thus, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no antihole of length greater than 4. □

Now, using Theorem 4.1 and the Strong Perfect Graph Theorem, we have everything needed to prove that $\Gamma(\mathbb{Z}_n)$ is perfect.

Theorem 5.2. *If $n = p^a q^b$ for distinct primes p, q and positive integers a, b, then $\Gamma(\mathbb{Z}_n)$ is perfect.*

Proof. By Lemmas 5.4 and 5.5, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole of odd length and no antihole of length greater than 4. This is equivalent to saying $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of odd length. Therefore, by the Strong Perfect Graph Theorem, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is perfect. Thus, by Theorem 4.1, $\Gamma(\mathbb{Z}_n)$ is perfect. □

5.3 $\Gamma(\mathbb{Z}_n)$ for $n = p^a qr$

Throughout this subsection, we assume $n = p^a qr$ for distinct primes p, q, r and positive integer a. We work with the type graph, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, whose vertices we denote with triples, as discussed in Section 3. Recall that the collection of vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is

$$\{T_d : d \text{ is a nontrivial divisor of } n\}.$$

Note that each nontrivial divisor of n is of the form $d = p^x q^y r^z$. For simplicity, we denote T_d with the triple (x, y, z), where $d = p^x q^y r^z$. Using this notation, we have

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_3) \text{ in } \Gamma^{\mathrm{T}}(\mathbb{Z}_n)$$

if and only if

$$x_1 + x_2 \geq a, \; y_1 + y_2 \geq 1, \text{ and } z_1 + z_2 \geq 1.$$

To prove that $\Gamma(\mathbb{Z}_n)$ is perfect, we first prove three lemmas.

Lemma 5.6. *With $n = p^a qr$ for distinct primes p, q, r and positive integer a, if H is a hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, then H has no vertex of the form $(v, 0, 0)$.*

Proof. Assume otherwise. Let H be either a hole or an antihole with vertex $(v, 0, 0)$. By Observation 5.5, there exists a path of length 5 contained in H and centered at $(v, 0, 0)$. Write the path as

$$(x_1, y_1, z_1) \sim (x_2, 1, 1) \sim (v, 0, 0) \sim (x_4, 1, 1) \sim (x_5, y_5, z_5)$$

with

$$(x_1, y_1, z_1) \nsim (x_4, 1, 1) \text{ and } (x_2, 1, 1) \nsim (x_5, y_5, z_5),$$

noting that the two neighbors of $(v, 0, 0)$ must have 1's as their second and third components since they are adjacent to $(v, 0, 0)$. Since $(x_2, 1, 1) \nsim (x_5, y_5, z_5)$ and $(x_4, 1, 1) \sim (x_5, y_5, z_5)$, we have $x_2 + x_5 < a \leq x_4 + x_5$, so $x_2 < x_4$. On the other hand, since $(x_1, y_1, z_1) \nsim (x_4, 1, 1)$ and $(x_1, y_1, z_1) \sim (x_2, 1, 1)$, we have $x_1 + x_4 < a \leq x_1 + x_2$, so $x_4 < x_2$. So, we have both $x_2 < x_4$ and $x_4 < x_2$, which is a contradiction. Thus, if H is a hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, then H has no vertex of the form $(v, 0, 0)$. $\square$

Lemma 5.7. *With $n = p^a qr$ for distinct primes p, q, r and positive integer a, if H is a hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, then H has no vertex of the form $(v, 1, 1)$.*

Proof. Assume otherwise. Let H be either a hole or an antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ with vertex $(v, 1, 1)$. We consider two cases.

Case 1. Assume $v \leq \frac{a}{2}$. By Observation 5.5, there exists a path of length 5 centered at $(v, 1, 1)$. Write the path as

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \sim (v, 1, 1) \sim (x_4, y_4, z_4) \sim (x_5, y_5, z_5)$$

with

$$(x_2, y_2, z_2) \nsim (x_4, y_4, z_4) \text{ and } (x_2, y_2, z_2) \nsim (x_5, y_5, z_5).$$

Since $(x_2, y_2, z_2) \sim (v, 1, 1)$, we have $x_2 + v \geq a$. It follows that $x_2 \geq \frac{a}{2}$ since $v \leq \frac{a}{2}$. Similarly, $x_4 \geq \frac{a}{2}$, so $x_2 + x_4 \geq a$. It then follows that (x_2, y_2, z_2) and (x_4, y_4, z_4) must each have a 0 in either their second or third component, but not both by Lemma 5.6. Without loss of generality, we have $(x_2, y_2, z_2) = (x_2, 0, 1)$ and $(x_4, y_4, z_4) = (x_4, 0, 1)$. Now, since $(x_1, y_1, z_1) \sim (x_2, 0, 1)$ and $(x_4, 0, 1) \sim (x_5, y_5, z_5)$, we must also have $(x_1, y_1, z_1) = (x_1, 1, z_1)$ and $(x_5, y_5, z_5) = (x_5, 1, z_5)$. Then $x_2 + x_5 < a \leq x_4 + x_5$ since $(x_2, 0, 1) \nsim (x_5, 1, z_5)$ and $(x_4, 0, 1) \sim (x_5, 1, z_5)$, so $x_2 < x_4$. On the other hand, $x_1 + x_4 < a \leq x_1 + x_2$ since $(x_1, 1, z_1) \nsim (x_4, 0, 1)$ and $(x_1, 1, z_1) \sim (x_2, 0, 1)$, so $x_4 < x_2$. Thus, we have both $x_2 < x_4$ and $x_4 < x_2$, which is a contradiction.

Case 2. Assume $v > \frac{a}{2}$. Using Observation 5.4, choose vertices (x_1, y_1, z_1) and (x_2, y_2, z_2) of H that are nonadjacent to $(v, 1, 1)$, but are adjacent to each other. Then since $(v, 1, 1) \nsim (x_1, y_1, z_1)$, we must have $v + x_1 < a$. Thus, since $v > \frac{a}{2}$, we have $x_1 < \frac{a}{2}$. Similarly, $x_2 < \frac{a}{2}$. Thus, $x_1 + x_2 < a$, which is a contradiction since $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$.

Thus, in both cases we have a contradiction, so if H is a hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, then H has no vertex of the form $(v, 1, 1)$. $\square$

Lemma 5.8. *If $n = p^a qr$ for distinct primes p, q, r and positive integer a, then there exists no hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.*

Proof. Assume H is a hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. Then by Lemmas 5.6 and 5.7, all vertices of H are of the form $(v, 1, 0)$ or $(v, 0, 1)$. Without loss of generality, assume $(v, 1, 0)$ is a vertex of H. Using Observation 5.5, there is a path of length 5 contained in H and centered at $(v, 1, 0)$ as follows:

$$(x_1, 1, 0) \sim (x_2, 0, 1) \sim (v, 1, 0) \sim (x_4, 0, 1) \sim (x_5, 1, 0)$$

with

$$(x_1, 1, 0) \nsim (x_4, 0, 1) \text{ and } (x_2, 0, 1) \nsim (x_5, 1, 0).$$

It follows that $x_1 + x_4 < a \leq x_1 + x_2$, while $x_2 + x_5 < a \leq x_4 + x_5$. Hence, we have both $x_4 < x_2$ and $x_2 < x_4$, which is a contradiction. Thus, there exists no hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. □

Now, using Theorem 4.1 and the Strong Perfect Graph Theorem, we have everything needed to prove that $\Gamma(\mathbb{Z}_n)$ is perfect.

Theorem 5.3. *If $n = p^a qr$ for distinct primes p, q, r and positive integer a, then $\Gamma(\mathbb{Z}_n)$ is perfect.*

Proof. By Lemma 5.8, we know that $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of length greater than 4. This is equivalent to saying $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of odd length. Therefore, by the Strong Perfect Graph Theorem, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is perfect. Thus, by Theorem 4.1, $\Gamma(\mathbb{Z}_n)$ is perfect. □

5.4 $\Gamma(\mathbb{Z}_n)$ for $n = pqrs$

Throughout this subsection, we assume $n = pqrs$ for distinct primes p, q, r, s. We work with the type graph, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$, whose vertices we denote with quadruples, as discussed in Section 3. Recall that the collection of vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is

$$\{T_d : d \text{ is a nontrivial divisor of } n\}.$$

Note that each nontrivial divisor of n is of the form $d = p^x q^y r^z s^w$, where each of x, y, z, and w is either 0 or 1. For simplicity, we denote T_d with the quadruple (x, y, z, w), where $d = p^x q^y r^z s^w$. Using this notation, we have

$$(x_1, y_1, z_1, w_1) \sim (x_2, y_2, z_2, w_2) \text{ in } \Gamma^{\mathrm{T}}(\mathbb{Z}_n)$$

if and only if

$$x_1 + x_2 \geq 1,\ y_1 + y_2 \geq 1,\ z_1 + z_2 \geq 1, \text{ and } w_1 + w_2 \geq 1.$$

To prove that $\Gamma(\mathbb{Z}_n)$ is perfect, we first prove a lemma.

Lemma 5.9. *If $n = pqrs$ for distinct primes p, q, r, s, then there exists no hole or antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$*

Proof. Assume otherwise. Let H be either a hole or an antihole of length greater than 4 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.

We first claim that H cannot contain any vertex with three 0-components. Without loss of generality, assume $(1, 0, 0, 0)$ is a vertex of H. Well, the vertex $(0, 1, 1, 1)$ is the only possible vertex that is a neighbor of $(1, 0, 0, 0)$. Hence it is not possible for H to contain a vertex with three 0-components because every vertex of H must have at least two neighbors.

Next, we claim that H contains no vertex with two 0-components and two 1-components. Without loss of generality, assume that $(1, 1, 0, 0)$ is a vertex of H. Using Observation 5.5, there is a path of length 5 contained in H and centered at $(1, 1, 0, 0)$:

$$(x_1, y_1, z_1, w_1) \sim (x_2, y_2, z_2, w_2) \sim (1, 1, 0, 0) \sim (x_4, y_4, z_4, w_4) \sim (x_5, y_5, z_5, w_5)$$

with

$$(x_1, y_1, z_1, w_1) \nsim (x_4, y_4, z_4, w_4) \text{ and } (x_2, y_2, z_2, w_2) \nsim (x_4, y_4, z_4, w_4).$$

Now, both neighbors of $(1, 1, 0, 0)$ in this path must have 1's in their last two components, and since they are nonadjacent to each other, they must each contain a 0 in a common component. The remaining component of these neighbors must not be equal, otherwise they would be the same vertex. Again, without loss of generality, we can assume the neighbors of $(1, 1, 0, 0)$ in the path are $(0, 0, 1, 1)$ and $(0, 1, 1, 1)$, like this:

$$(x_1, y_1, z_1, w_1) \sim (0, 0, 1, 1) \sim (1, 1, 0, 0) \sim (0, 1, 1, 1) \sim (x_5, y_5, z_5, w_5).$$

Now, we see that the first vertex in this path must have 1's in its first two components since it is adjacent to $(0, 0, 1, 1)$. Thus, we have the path

$$(1, 1, z_1, w_1) \sim (0, 0, 1, 1) \sim (1, 1, 0, 0) \sim (0, 1, 1, 1) \sim (x_5, y_5, z_5, w_5).$$

However, this is not possible since the the first and fourth vertices of this path must not be adjacent. Thus, we have a contradiction, so we know that H cannot contain a vertex with two 0-components and two 1-components.

The only possibility remaining is for H to consist only of vertices with three 1-components and one 0-component. However, there are only four such quadruples. Since H must have at least five vertices, this is not possible.

Thus, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of length greater than 4. $\square$

Now, using Theorem 4.1 and the Strong Perfect Graph Theorem, we have everything needed to prove that $\Gamma(\mathbb{Z}_n)$ is perfect.

Theorem 5.4. *If $n = pqrs$ for distinct primes p, q, r, s, then $\Gamma(\mathbb{Z}_n)$ is perfect.*

Proof. By Lemma 5.9 we know that $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of length greater than 4. This is equivalent to saying $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains no hole or antihole of odd length. Therefore, by the Strong Perfect Graph Theorem, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is perfect. Thus, by Theorem 4.1, $\Gamma(\mathbb{Z}_n)$ is perfect. $\square$

5.5 Perfect n

Throughout this section, we have proven that for certain values of n, $\Gamma(\mathbb{Z}_n)$ is perfect. This was done through Theorems 5.1–5.4, which can be summarized by the following theorem.

Theorem 5.5. *For any n in the following cases, $\Gamma(\mathbb{Z}_n)$ is perfect:*

$$\begin{aligned}
&n = p^a \textit{ for a prime } p \textit{ and positive integer } a,\\
&n = p^a q^b \textit{ for distinct primes } p, q \textit{ and positive integers } a, b,\\
&n = p^a qr \textit{ for distinct primes } p, q, r \textit{ and positive integer } a, \textit{ and}\\
&n = pqrs \textit{ for distinct primes } p, q, r, s.
\end{aligned}$$

We note that this leaves only the following three cases:

$$\begin{aligned}
&n = p^{1+a} q^{1+b} r^c \text{ for distinct primes } p, q, r \text{ and positive integers } a, b, c,\\
&n = p^{1+a} q^b r^c s^d \text{ for distinct primes } p, q, r, s \text{ and positive integers } a, b, c, d, \text{ and}\\
&n = p_1^{k_1} p_2^{k_2} \ldots p_l^{k_l} \text{ for distinct primes } p_1, p_2, \ldots, p_l \text{ and positive integers } k_1, k_2, \ldots, k_l,\\
&\qquad\qquad\qquad \text{where } l \geq 5.
\end{aligned}$$

In the following section, we show that $\Gamma(\mathbb{Z}_n)$ is not perfect in any of these remaining cases.

6 Values of n for which $\Gamma(\mathbb{Z}_n)$ is not Perfect

In this section, we will show that $\Gamma(\mathbb{Z}_n)$ is not perfect for the values of n which were *not* considered in the preceding section. These values of n are covered by the three cases in the following lemma.

Lemma 6.1. *For any n listed in the following cases, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains a hole of length 5:*

$$\begin{aligned}
&n = p^{1+a} q^{1+b} r^c \textit{ for distinct primes } p, q, r \textit{ and positive integers } a, b, c,\\
&n = p^{1+a} q^b r^c s^d \textit{ for distinct primes } p, q, r, s \textit{ and positive integers } a, b, c, d, \textit{ and}\\
&n = p_1^{k_1} p_2^{k_2} \ldots p_l^{k_l} \textit{ for distinct primes } p_1, p_2, \ldots, p_l \textit{ and positive integers } k_1, k_2, \ldots, k_l,\\
&\qquad\qquad\qquad \textit{where } l \geq 5.
\end{aligned}$$

Proof. For each case, we illustrate a hole of length 5 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.

Case 1. Assume $n = p^{1+a} q^{1+b} r^c$ for distinct primes p, q, r and positive integers a, b, c. Using the tuple notation analogous to that used throughout Section 5, vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ are represented by triples, where

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \text{ in } \Gamma^{\mathrm{T}}(\mathbb{Z}_n)$$

if and only if

$$x_1 + x_2 \geq 1 + a, \ y_1 + y_2 \geq 1 + b, \text{ and } z_1 + z_2 \geq c.$$

One can easily check that the figure shown below is a hole in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.

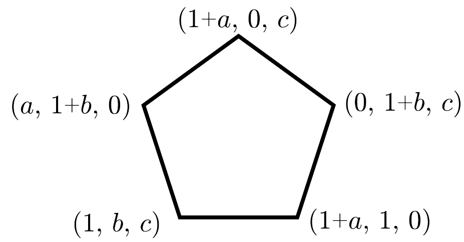

Figure 6.1: A hole of length 5 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ where $n = p^{1+a}q^{1+b}r^c$ for distinct primes p, q, r and positive integers a, b, c.

Case 2. Assume $n = p^{1+a}q^br^cs^d$ for distinct primes p, q, r, s and positive integers a, b, c, d. In this case, vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ are represented by quadruples, where

$$(x_1, y_1, z_1, w_1) \sim (x_2, y_2, z_2, w_2) \text{ in } \Gamma^{\mathrm{T}}(\mathbb{Z}_n)$$

if and only if

$$x_1 + x_2 \geq 1 + a,\ \ y_1 + y_2 \geq b,\ \ z_1 + z_2 \geq c, \text{ and } w_1 + w_2 \geq d.$$

One can easily check that the figure shown below is a hole in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.

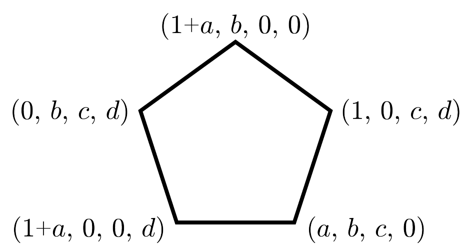

Figure 6.2: A hole of length 5 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ where $n = p^{1+a}q^br^cs^d$ for distinct primes p, q, r, s and positive integers a, b, c, d.

Case 3. Assume $n = p_1^{k_1}p_2^{k_2}\cdots p_l^{k_l}$ for distinct primes $p_1, p_2, \ldots, p_l$ and positive integers $k_1, k_2, \ldots, k_l$, where $l \geq 5$. In this case, vertices of $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ are represented by l-tuples, where two of these l-tuples are adjacent if and only if the i^{th} components sum to at least k_i for all $i \in \{1, 2, \ldots, l\}$. Here, we specify five vertices as follows, noting that the first five components in each l-tuple are carefully chosen. If l exceeds 5, the remaining components are simply filled with the full exponent values $k_6, \ldots, k_l$.

$$\begin{aligned}
v_1 &= (k_1, k_2, k_3, 0, 0, k_6, \ldots, k_l)\\
v_2 &= (k_1, 0, 0, k_4, k_5, k_6, \ldots, k_l)\\
v_3 &= (0, k_2, k_3, k_4, 0, k_6, \ldots, k_l)\\
v_4 &= (k_1, 0, k_3, 0, k_5, k_6, \ldots, k_l)\\
v_5 &= (0, k_2, 0, k_4, k_5, k_6, \ldots, k_l)
\end{aligned}$$

Given these vertices, one can easily check that the figure shown below is a hole in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$.

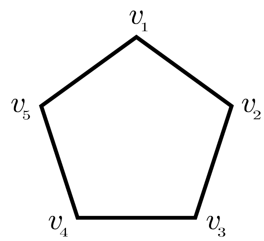

Figure 6.3: A hole of length 5 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ where $n = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l}$ for distinct primes $p_1, p_2, \ldots, p_l$ and positive integers $k_1, k_2, \ldots, k_l$, with $l \geq 5$.

In each case, we have illustrated a hole of length 5 in $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$. $\square$

Theorem 6.1. *For any n in the following cases, $\Gamma(\mathbb{Z}_n)$ is not perfect:*

$n = p^{1+a} q^{1+b} r^c$ *for distinct primes p, q, r and positive integers a, b, c,*
$n = p^{1+a} q^b r^c s^d$ *for distinct primes p, q, r, s and positive integers a, b, c, d, and*
$n = p_1^{k_1} p_2^{k_2} \ldots p_l^{k_l}$ *for distinct primes $p_1, p_2, \ldots, p_l$ and positive integers $k_1, k_2, \ldots, k_l$,*
where $l \geq 5$.

Proof. For each case, Lemma 6.1 shows that $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ contains a hole of length 5. Therefore, by the Strong Perfect Graph Theorem, $\Gamma^{\mathrm{T}}(\mathbb{Z}_n)$ is not perfect. Thus, by Theorem 4.1, $\Gamma(\mathbb{Z}_n)$ is not perfect. $\square$

7 Summary

Theorems 5.5 and 6.1 together classify all values of n according to whether $\Gamma(\mathbb{Z}_n)$ is perfect or not. The classification depends on the prime factorization of n and can be summarized by our main theorem.

Main Theorem. *The zero-divisor graph of $\mathbb{Z}_n$, $\Gamma(\mathbb{Z}_n)$, is perfect if and only if n has one of the following four forms.*

1. *$n = p^a$ for a prime p and positive integer a*
2. *$n = p^a q^b$ for distinct primes p, q and positive integers a, b*
3. *$n = p^a qr$ for distinct primes p, q, r and positive integer a*
4. *$n = pqrs$ for distinct primes p, q, r, s*

This means that for n with at most two distinct prime factors, $\Gamma(\mathbb{Z}_n)$ is perfect. Also, for n with at least five distinct prime factors, $\Gamma(\mathbb{Z}_n)$ is not perfect. Finally, for n with either three or four distinct prime factors, $\Gamma(\mathbb{Z}_n)$ is perfect only in the cases where $n = p^a qr$ or $n = pqrs$ for distinct primes p, q, r, s and positive integer a.

Further work might be done to see if the techniques we use in this paper could be adjusted to determine the perfectness of zero-divisor graphs over rings other than $\mathbb{Z}_n$. For example, one might consider zero-divisor graphs of the rings of diagonal matrices over $\mathbb{Z}_n$.

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