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Iterated Perpendicular Constructions from Interior Points on N-gons

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Two years ago I came across a very beautiful geometric construction (Figure 1), on a tee-shirt I received from the Upstate New York math team. Written on the shirt was a surprising result of iterating a certain construction multiple times. The coaches told me that the proof for the result which they had was very long and very complex. Such a beautiful problem deserves a beautiful answer. So, I came up with a very nice proof which took less than a minute to present to the team the following year. Since then I have delved further into the construction, arriving at some very interesting results. This paper proves some of those results which arise from iterating the geometric construction. For the purpose of this paper P(n,x) will denote a polygon with n sides (an n-gon) after iterating the construction x times. By iterating the construction I mean taking the resulting polygon created by a construction and applying the construction to that resulting polygon.

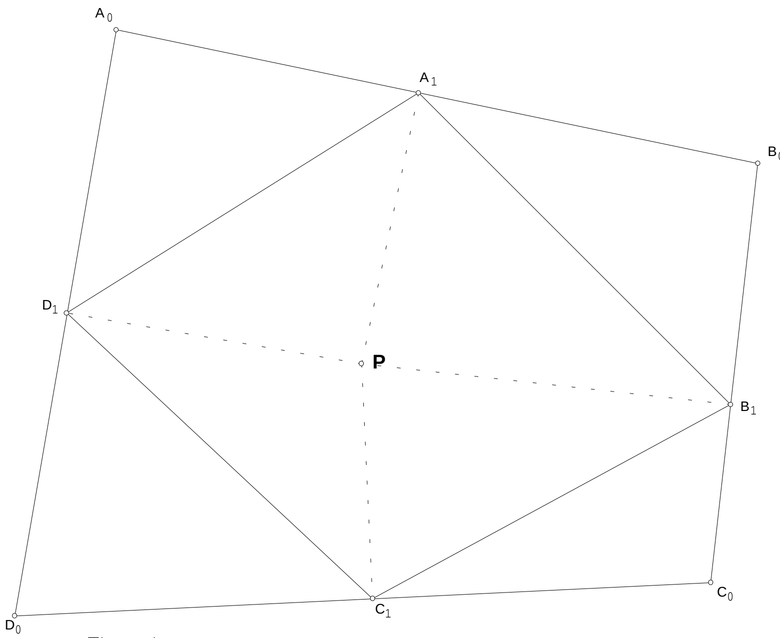

Figure 1

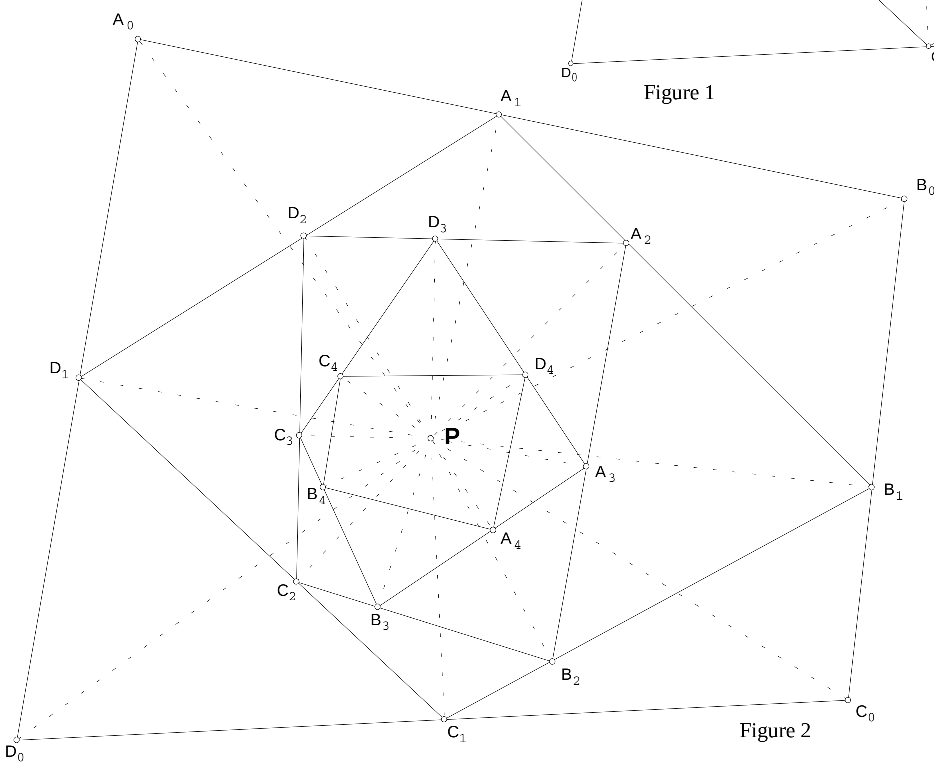

Figure 2

The construction: Convex quadrilateral $A_0B_0C_0D_0$ is given, and a point P is placed in the interior of $A_0B_0C_0D_0$ such that perpendiculars can be drawn to all sides. From point P to the four sides of quadrilateral $A_0B_0C_0D_0$ lines perpendicular to the sides are drawn. The intersection points with the sides of $A_0B_0C_0D_0$ are labeled A_1, B_1, C_1, D_1, and are connected to form quadrilateral $A_1B_1C_1D_1$ (Figure 1). The same process using interior point P is repeated now with quadrilateral $A_1B_1C_1D_1$ to form quadrilateral $A_2B_2C_2D_2$. After twice repeating the construction, $A_4B_4C_4D_4$ is created (Figure 2).

Now that we understand the construction and iteration process, we can investigate some interesting results.

Theorem 1: The initial quadrilateral $A_0B_0C_0D_0$ has equal angles to quadrilateral $A_4B_4C_4D_4$.

First we will look at quadrilateral $D_1A_0A_1P$. Observe that this quadrilateral is cyclic (Figure 2 provides a proof of this and an explanation for cyclic quadrilaterals is found below).

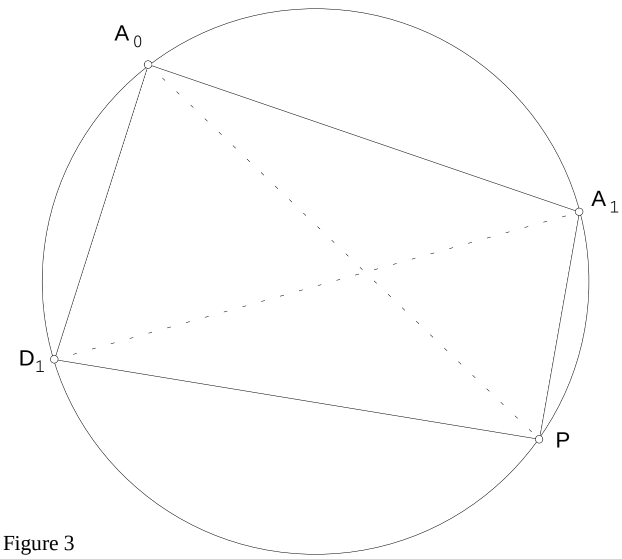

Figure 3

A quadrilateral which is cyclic can be inscribed in a circle. A quadrilateral is cyclic if opposite angles sum to 180 degrees. Since segment PD_1 is perpendicular to segment A_0D_1 and since segment PA_1 is perpendicular to segment A_0A_1, quadrilateral $A_0A_1PD_1$ is cyclic (as shown in figure 3).

Without loss of generalization we will just be dealing with $\angle B_0A_0D_0$, of the initial quadrilateral. So, $\angle B_0A_0D_0$ is composed of two angles which are $\angle B_0A_0P$ and $\angle D_0A_0P$. Now, since $D_1A_0A_1P$ is a cyclic quadrilateral, it has the property that $\angle A_1A_0P$ (which can also be called $\angle B_0A_0P$) = $\angle A_1D_1P$. This occurs because both angles subtend the same arc and both have vertices which lie on the circle's circumference. This shows why $\angle B_0A_0P$ of the initial quadrilateral is equal to $\angle A_1D_1P$ of the first constructed quadrilateral. The same argument shows that $\angle D_1A_0P$ (which can also be called $\angle D_0A_0P$) = $\angle D_1A_1P$.

Now, we apply the same reasoning for quadrilateral $A_1B_1C_1D_1$ and for each subsequently formed quadrilateral. If we look at how a given angle from the initial quadrilateral moves to each subsequently formed quadrilateral we see that $\angle D_0A_0P = \angle D_1A_1P = \angle D_2A_2P = \angle D_3A_3P = \angle D_4A_4P$. The same applies for $\angle B_0A_0P = \angle A_1D_1P = \angle D_2C_2P = \angle C_3B_3P = \angle B_4A_4P$. The basic idea here is that the angles which constitute the initial angle $\angle B_0A_0D_0$ move around the subsequently formed quadrilaterals and come back together after four constructions to form $\angle B_4A_4D_4$. Had we chosen any other initial angle, the same result would hold. Since each angle which is in the initial quadrilateral is reformed in the fourth construction, and since the angles are in the same order, the quadrilaterals are similar. Therefore, $A_0B_0C_0D_0$ has congruent angles to $A_4B_4C_4D_4$. □

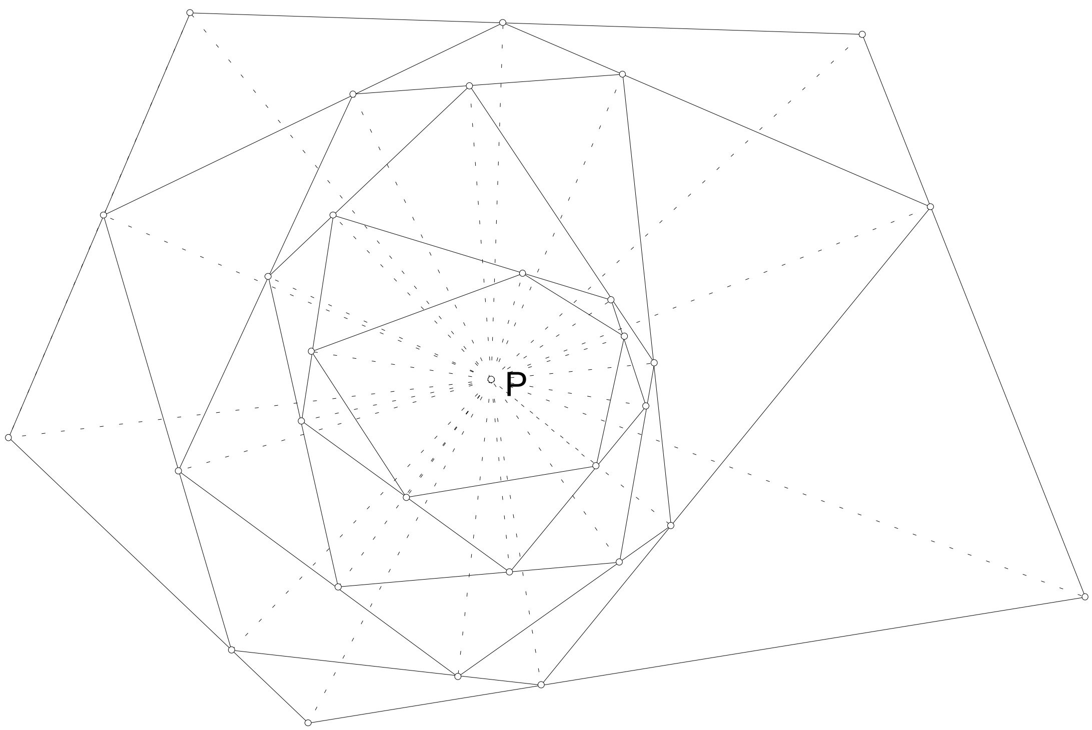

Figure 3

Theorem 2: An initial polygon with n sides, denoted as P(n,$_0$), has equal angles to the polygon P(n,$_n$) which is created by iterating the previously discussed construction n times. This can be also written as polygon $A_0B_0C_0$.....Z_0..... has equal angles as polygon $A_nB_nC_n$.....Z_n.....

The proof for Theorem 2 follows in the same manner as Theorem 1's proof. Every angle in the initial quadrilateral, P(n,$_0$), are divided into two angles by the segment from the point P to the vertex of the quadrilateral. Like in the quadrilateral, the perpendiculars which are extended to each side of the polygon create cyclic quadrilaterals. In the same manner as with the quadrilateral, the two angles which constituted the initial polygon's angle, move in opposite directions around each subsequently formed polygon through each construction. In the quadrilateral there were eight different angles in each quadrilateral (each of the quadrilateral's angles was split by the segment to *P*). Generally there are twice as many angles of that manner as there are sides. So, since there are 2n positions for angles, and each of the two angles moves through one position every time the construction is iterated, it takes n iterations of the construction until the split angles reform to create the same angle as the initial polygon had. This holds true for each set of two angles, and therefore all of the polygon's angles, after n iterated construction, will be the same as the initial polygon's angles. This implies that the two polygons have equal angles or that $P(n,_0)$ has equal angles as $P(n,_n)$.

Now that we have examined the angles which result from iterative constructions, we should look at the side lengths.

Theorem 3: The ratio of the length of a given side of $P(n,_0)$ to the length of the analogous side of $P(n,_n)$ is constant and equal to the product of the sin's of each angle in polygon $P(n,_0)$, or $\sin(\angle B_0A_0P) \times \sin(\angle C_0B_0P) \times \ldots\ldots \times \sin(\angle Z_0Y_0P) \times \ldots\ldots$

Let us take any edge of $P(n,_0)$, say A_0B_0. That is a side of the triangle A_0B_0P, formed by connecting points A_0 and B_0 to point P. Triangle A_0B_0P is similar to triangle A_nB_nP because they contain the same angles(this is sufficient to show similarity for triangles). So, it is sufficient to find the ratio, which we will call r, of any side of triangle A_0B_0P to the similar side on triangle A_nB_nP. Since the ratio of the length of segment B_0P to the length of segment B_nP is equal to r (the two lengths are in the similar triangles), triangle B_0C_0P is not only similar to triangle B_0C_0P, but also has the same ratio r as the first triangle. Using the same argument, it can be shown that the ratio of the length of any given side of $P(n,_0)$ to the analogous side in $P(n,_n)$ is the same, and is equal to *r*.

With out loss of generality, we can now select segment A_0P from triangle A_0B_0P. That segment is the hypotenuse of the right triangle A_0A_1P. So, the product of the length of segment A_0P and the sin of angle B_0A_0P is, by right triangle trigonometry, the length of segment A_1P. Now we can apply the same idea for $P(n,_1)$, or the first constructed polygon. Segment A_1P is the hypotenuse of the right triangle A_1A_2P. So, the product of the length of segment A_1P and the sin of angle B_1A_1P is, by right triangle trigonometry, the length of segment A_2P. This continues until the length of segment A_nP is arrived at. That length will be equal to $A_{n-1}P \text{ x } sin(B_{n-1}A_{n-1}P)$. Observe that for a given iteration *i*, the length of segment A_iP is $A_{i-1}P \text{ x } sin(B_{i-1}A_{i-1}P)$. So, working backward for A_nP we see that the length of segment $A_nP = A_0P \text{ x } sin(B_0A_0P) \text{ x } sin(B_1A_1P) \text{ x ..... x } sin(B_{n-1}A_{n-1}P)$. Also we know that $\angle B_1A_1P$ is equal to $\angle C_0B_0P$. This pattern continues and $\angle B_2A_2P = \angle D_0C_0P$ and so on. So, by substituting in the equivalent angles from $P(n,_0)$ into the equation for the length of segment A_nP, we find that the segment's length is $A_0P \text{ x } sin(B_0A_0P) \text{ x } sin(C_0B_0P) \text{ x ..... x } sin(Z_0Y_0P) \text{ x......}$ Therefore, the ratio of the length of a given side of $P(n,_0)$ to the length of the analogous side of $P(n,_n)$ is equal to the product of the sin's of each angle in polygon $P(n,_0)$, or $sin(\angle B_0A_0P) \text{ x } sin(\angle C_0B_0P) \text{ x ...... x } sin(\angle Z_0Y_0P) \text{ x ......}$

Corollary 1: $P(n,_0) \sim P(n,_n)$, or the initial polygon is similar to the polygon which is created by iterating the given construction n times.

Theorem 2 proves that all angles in the two polygons are equal and in the same order and Theorem 3 proves that the ratio between lengths of all the analogous sides of the polygons is equal.

The next question to ask is about the orientation of polygon $P(n,_n)$.

Theorem 4: The polygon P(n,$_n$) has rotated $90n - \angle B_0A_0P - \angle C_0B_0P - .... - \angle Z_0Y_0P -....$ degrees around P, from polygon P(n,$_0$).

We will first see that this holds true for our original quadrilateral where n=4. It suffices for the proof to show that a single segment, say segment A_0P rotates the given degree amount, because each other segment rotates the same amount. So, segment A_0P initially rotates through $\angle A_0PA_1$ until it lies on segment A_1P. Since triangle A_0A_1P is a right triangle $\angle A_0PA_1$ is equal to $90° - \angle A_1A_0P = 90° - \angle B_0A_0P$ (we get this by extending the end point from A_1 to B_0). Now, we continue for segment A_1P, which rotates $90° - \angle B_1A_1P$ until it lies on segment A_2P. Segment A_2P rotates a measure of $90° - \angle B_2A_2P$ until segment A_3P, which then rotates an additional measure of $90° - \angle B_3A_3P$. So, the segment A_0P rotated a total angle of $90° - \angle B_0A_0P + 90° - \angle B_1A_1P + 90° - \angle B_2A_2P + 90° - \angle B_3A_3P$. Lastly, using the same angle substitution as Theorem 3 used, $\angle B_1A_1P = \angle C_0B_0P$, and $\angle B_2A_2P = \angle D_0C_0P$, and $\angle B_3A_3P = \angle A_0D_0P$. So, the angle which segment A_0P is rotated is actually $360° - \angle B_0A_0P - \angle C_0B_0P - \angle D_0C_0P - \angle A_0D_0P$.

For P(n,$_n$), there are n different rotations which segment A_0P goes through until it reaches A_nP. As for the quadrilateral, the total rotation is $90° - \angle B_0A_0P + 90° - \angle B_1A_1P +.....+ 90° - \angle B_{n-1}A_{n-1}P$. Using the same angle substitution as used above and as used in Theorem 3, this total rotation is equal to $90n° - \angle B_0A_0P - \angle C_0B_0P -....- \angle Z_0Y_0P-.....$

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