Rose-Hulman Undergraduate Mathematics Journal

Volume 3
Issue 1

Article 4

Three Term Identities for the Coefficients of Certain Infinite Products

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THREE TERM IDENTITIES FOR THE COEFFICIENTS OF CERTAIN INFINITE PRODUCTS

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1. Introduction

Let N be an integer. For non-negative integers n, let $P_N(n)$ be the coefficients of the series defined by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^N} = \sum_{n=0}^{\infty} P_N(n) q^n.$$

and let $P_N(r) = 0$ for a rational number r not equal to any non-negative integer.

Recently Farkas and Kra presented five three-term identities for the coefficients $P_N(n)$.

Theorem 1[F-K]**.** *For all non-negative integers n, we have*

$$P_{-24}(2n+1) = -2^{11} P_{-24}\left(\frac{n-1}{2}\right) - 2^3 \cdot 3 P_{-24}(n), \tag{1.1}$$

$$P_{-12}(3n+1) = -3^5 P_{-12}\left(\frac{n-1}{3}\right) - 2^2 \cdot 3 P_{-12}(n), \tag{1.2}$$

$$P_{-6}(5n+1) = -5^2 P_{-6}\left(\frac{n-1}{5}\right) - 2 \cdot 3 P_{-6}(n), \tag{1.3}$$

$$P_{-4}(7n+1) = -7 P_{-4}\left(\frac{n-1}{7}\right) - 2^2 P_{-4}(n), \tag{1.4}$$

$$P_{-2}(13n+1) = -P_{-2}\left(\frac{n-1}{13}\right) - 2 P_{-2}(n). \tag{1.5}$$

Also in the paper, Farkas and Kra mentioned that Mordell [M] proved that for all primes l and all positive integers n, we have

$$P_{-24}(ln-1) = P_{-24}(l-1) P_{-24}(n-1) - l^{11} P_{-24}\left(\frac{n}{l} - 1\right). \tag{1.6}$$

It is clear that (1.1) is a special case of (1.6) with $l = 2$ and n replaced by $n+1$. The authors state that "it is not at all clear whether (1.2), (1.3), (1.4) and (1.5) are also special cases of more general identities." The purpose of this paper is to show that each of the

The author thanks The Colgate Division of Natural Sciences and Mathematics for its generous support.

above-mentioned three-term identities is just one of an infinite family of identities. Recall that $\left(\frac{a}{p}\right)$ is the Legendre symbol defined for primes p and integers a as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \text{ and } p \nmid a, \\ 0 & \text{if } p \mid a, \\ -1 & \text{otherwise.} \end{cases}$$

Our main result is

Theorem 2. *For all primes l and all positive integers n, we have*

$$P_{-12}\left(\frac{ln-1}{2}\right) = P_{-12}\left(\frac{l-1}{2}\right) P_{-12}\left(\frac{n-1}{2}\right) - \left(\frac{4}{l}\right) l^5 P_{-12}\left(\frac{\frac{n}{l}-1}{2}\right), \tag{1.7}$$

$$P_{-6}\left(\frac{ln-1}{4}\right) = P_{-6}\left(\frac{l-1}{4}\right) P_{-6}\left(\frac{n-1}{4}\right) - \left(\frac{-4}{l}\right) l^2 P_{-6}\left(\frac{\frac{n}{l}-1}{4}\right), \tag{1.8}$$

$$P_{-4}\left(\frac{ln-1}{6}\right) = P_{-4}\left(\frac{l-1}{6}\right) P_{-4}\left(\frac{n-1}{6}\right) - \left(\frac{36}{l}\right) l P_{-4}\left(\frac{\frac{n}{l}-1}{6}\right), \tag{1.9}$$

$$P_{-2}\left(\frac{ln-1}{12}\right) = P_{-2}\left(\frac{l-1}{12}\right) P_{-2}\left(\frac{n-1}{12}\right) - \left(\frac{-36}{l}\right) P_{-2}\left(\frac{\frac{n}{l}-1}{12}\right). \tag{1.10}$$

From Theorem 2, it follows that

(i) our technique also gives a proof of (1.6),

(ii) the statements in (1.2)-(1.5) are all special cases of Theorem 2. For example, if we let $l = 5$ and replace n with $4n + 1$ in (1.8), we get (1.3).

The next corollary follows from Theorem 2.

Corollary 1. *For all primes l and all positive integers n,*

$$\text{if } l \not\equiv 1 \pmod 4, \text{ then } \quad P_{-6}\left(\frac{ln-1}{4}\right) = -\left(\frac{-4}{l}\right) l^2 P_{-6}\left(\frac{\frac{n}{l}-1}{4}\right),$$

$$\text{if } l \not\equiv 1 \pmod 6, \text{ then } \quad P_{-4}\left(\frac{ln-1}{6}\right) = -\left(\frac{36}{l}\right) l P_{-4}\left(\frac{\frac{n}{l}-1}{6}\right),$$

$$\text{if } l \not\equiv 1 \pmod{12}, \text{ then } \quad P_{-2}\left(\frac{ln-1}{12}\right) = -\left(\frac{-36}{l}\right) P_{-2}\left(\frac{\frac{n}{l}-1}{12}\right).$$

2. Preliminaries

We now briefly introduce modular forms. For complete details, see [K]. Let $\mathbb{H}$ denote the upper half of the complex plane, $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The set $\mathrm{SL}_2(\mathbb{Z})$ is defined as $\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1; a, b, c, d \in \mathbb{Z} \right\}$ and is sometimes denoted by Γ. Let N be a positive integer. One important subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod N \right\}.$$

Given an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define the following transformation for $z \in \mathbb{H}$: $\gamma z := \frac{az+b}{cz+d}$.

Let f be a holomorphic function on the upper half-plane $\mathbb{H}$, and let k be an integer. If γ is a 2×2 matrix with rational entries and positive determinant, then we define $f(z) \mid [\gamma]_k$ as

$$f(z) \mid [\gamma]_k := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

It is easy to verify that

$$f(z) \mid [\gamma_1\gamma_2]_k = (f(z) \mid [\gamma_1]_k) \mid [\gamma_2]_k. \tag{2.1}$$

Let $f(z)$ be a holomorphic function on $\mathbb{H}$. Let k be an integer and N be a positive integer. Recall that a *Dirichlet character mod N* is a function $\chi(n) : \mathbb{Z} \to \mathbb{C}$, not identically zero, which satisfies, for all integers n and m,

(i) $\chi(n) = \chi(m)$ if $n \equiv m \pmod{N}$,
(ii) $\chi(n) = 0$ if $\gcd(n, N) > 1$,
(iii) $\chi(nm) = \chi(n)\chi(m)$.

Let χ be a Dirichlet character mod N. Then $f(z)$ is called a *modular form of weight k for $\Gamma_0(N)$ with character χ* if

$$f(z) \mid [\gamma]_k = \chi(d) f(z) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tag{2.2}$$

and if for any $\gamma_0 \in \Gamma$,

$$f(z) \mid [\gamma_0]_k \text{ has a Fourier expansion of the form } \sum_{n=0}^{\infty} a(n) q_N^n, \quad \text{where } q_N := e^{2\pi i z/N}. \tag{2.3}$$

A modular form $f(z)$ is called a *cusp form* if in addition we have

$$a(0) = 0 \;\text{ in (2.3) for all } \gamma_0 \in \Gamma. \tag{2.4}$$

The set of such modular forms is denoted $M_k(\Gamma_0(N), \chi)$, and the set of cusp forms is denoted $S_k(\Gamma_0(N), \chi)$. Every modular form has a Fourier expansion at infinity,

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n, \qquad \text{where } q := e^{2\pi i z}.$$

We will identify f with its expansion.

Note that the conditions (2.2), (2.3) and (2.4) are preserved under addition and scalar multiplication, so the sets of modular forms and the sets of cusp-forms of fixed weight, character and N are complex vector spaces.

Another important ingredient in our study are the Hecke operators. For each space $M_k(\Gamma_0(N),\chi)$, there exists a family of Hecke operators $T(p)$, one operator for each prime p. These are linear operators which preserve the space of cusp forms:

$$T(p): S_k(\Gamma_0(N),\chi) \to S_k(\Gamma_0(N),\chi). \tag{2.5}$$

They can be defined explicitly as follows: If $f(z)=\sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N),\chi)$, then

$$f(z) \mid T(p) := \sum_{n=0}^{\infty} (a(pn)+\chi(p)p^{k-1}a(n/p))q^n, \quad \text{where} \quad a(n/p)=0 \text{ if } p \nmid n.$$

A modular form $f(z)$ is called a *primitive eigenform* if for every prime p, there exists a scalar λ_p such that

$$f(z) \mid T(p) = \lambda_p f(z). \tag{2.6}$$

3. Proof of Theorem 2

We first establish the relationship between the coefficients of modular forms and the coefficients of the functions $P_N(n)$. We will use Dedekind's η-function,

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \qquad q = e^{2\pi i z}.$$

For positive integers n, define integers $a(n), b(n), c(n)$ and $d(n)$ by

$$\sum_{n=1}^{\infty} a(n)q^n := \eta^{12}(2z) = q\prod_{n=1}^{\infty}(1-q^{2n})^{12} = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + \cdots,$$

$$\sum_{n=1}^{\infty} b(n)q^n := \eta^{6}(4z) = q\prod_{n=1}^{\infty}(1-q^{4n})^{6} = q - 6q^5 + 9q^9 + 10q^{13} - 30q^{17} + \cdots,$$

$$\sum_{n=1}^{\infty} c(n)q^n := \eta^{4}(6z) = q\prod_{n=1}^{\infty}(1-q^{6n})^{4} = q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} + \cdots,$$

$$\sum_{n=1}^{\infty} d(n)q^n := \eta^{2}(12z) = q\prod_{n=1}^{\infty}(1-q^{12n})^{2} = q - 2q^{13} - q^{25} + 2q^{37} + q^{49} + \cdots.$$

We now state a Theorem that implies Theorem 2.

Theorem 3. *For all primes l and positive integers n, we have*

$$a(ln) = a(l)a(n) - l^5\left(\frac{4}{l}\right)a\left(\frac{n}{l}\right), \tag{3.1}$$

$$b(ln) = b(l)b(n) - l^2\left(\frac{-4}{l}\right)b\left(\frac{n}{l}\right), \tag{3.2}$$

$$c(ln) = c(l)c(n) - l\left(\frac{36}{l}\right)c\left(\frac{n}{l}\right), \tag{3.3}$$

$$d(ln) = d(l)d(n) - \left(\frac{-36}{l}\right)d\left(\frac{n}{l}\right). \tag{3.4}$$

The following Lemma shows that Theorem 3 implies Theorem 2.

Lemma 1. *For all positive integers n, we have*

$$\begin{aligned} a(n) &= P_{-12}\left(\frac{n-1}{2}\right), \\ b(n) &= P_{-6}\left(\frac{n-1}{4}\right), \\ c(n) &= P_{-4}\left(\frac{n-1}{6}\right), \\ d(n) &= P_{-2}\left(\frac{n-1}{12}\right). \end{aligned}$$

Proof of Lemma 1. By the definitions of $P_N(n)$ and $a(n)$, we have

$$\eta^{12}(2z) = q\prod_{n=1}^{\infty}(1-q^{2n})^{12} = q\sum_{n=0}^{\infty}P_{-12}(n)q^{2n} = \sum_{n=1}^{\infty}a(n)q^n.$$

Therefore, we get $a(n) = P_{-12}\left(\frac{n-1}{2}\right)$. The same technique can be applied to $\eta^6(4z), \eta^4(6z)$ and $\eta^2(12z)$ and the other three statements in Lemma 1 can be proved similarly. $\square$

We now define a family of characters which will be used in our proofs. If $4 \mid N$, then define χ_N by

$$\chi_N(d) := \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } \gcd(d,N)=1, \\ 0 & \text{if } \gcd(d,N) \neq 1. \end{cases}$$

We also define the trivial character mod N as the following:

$$\chi_N^{\text{triv}}(d) := \begin{cases} 1 & \text{if } \gcd(d,N)=1, \\ 0 & \text{if } \gcd(d,N) \neq 1. \end{cases}$$

Clearly, χ_N (as well as χ_N^{triv}) is a Dirichlet character mod N.

To prove Theorem 3, we need the following two Lemmas:

Lemma 2.

$$\begin{aligned} &\eta^{12}(2z) \in S_6\left(\Gamma_0(4), \chi_4^{\text{triv}}\right), \\ &\eta^{6}(4z) \in S_3\left(\Gamma_0(16), \chi_{16}\right), \\ &\eta^{4}(6z) \in S_2\left(\Gamma_0(36), \chi_{36}^{\text{triv}}\right), \\ &\eta^{2}(12z) \in S_1\left(\Gamma_0(144), \chi_{144}\right). \end{aligned}$$

Lemma 3. *All four functions in Lemma 2 are primitive eigenforms.*

Proof of Lemma 2. Let $f(z) = \eta^6(4z)$. By the definition of cusp-forms, we have to show that with $N = 16$ and $\chi = \chi_{16}$, the statements in (2.2), (2.3) and (2.4) hold. We will use the famous transformation formula for the η-function: (see, for example, [D-K-M] or [R]):

$$\text{If } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \text{ and } c \text{ is positive, then } \eta(\gamma z) = \varepsilon(a,b,c,d)(cz+d)^{1/2}\eta(z),$$

$$\text{where} \quad \varepsilon(a,b,c,d) := \begin{cases} \pm\exp\left(\frac{2\pi i}{24}(-3c - bd(c^2-1) + c(a+d))\right) & \text{if } c \text{ is odd,} \\ \pm\exp\left(\frac{2\pi i}{24}(3d - 3 - ac(d^2-1) + d(b-c))\right) & \text{if } d \text{ is odd.} \end{cases}$$

We remark that in the definition of ε, $\pm$ is explicitly determined, but the value is not important for our purpose.

Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(16)$, and $c > 0$. Then d is odd, and we have

$$\begin{aligned} f(\gamma z) &= \eta^6(4\gamma z) \\ &= \eta^6\left(\frac{a\cdot 4z + 4b}{(c/4)(4z) + d}\right) \\ &= \varepsilon^6(a, 4b, c/4, d)(cz+d)^3\eta^6(4z), \end{aligned} \tag{3.5}$$

where

$$\varepsilon^6(a,4b,c/4,d) = \exp\left(\frac{\pi i}{2}(3d - 3 - c/4\cdot a(d^2-1) + 4b\cdot d - d\cdot c/4)\right) = (-1)^{\frac{d-1}{2}} \tag{3.6}$$

That is, we get $\varepsilon^6(a,4b,c/4,d) = \chi_{16}(d)$. Using (3.5) and (3.6), we have verified (2.2) when $c > 0$:

$$\text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(16), \quad \text{then } f(z) \mid [\gamma]_3 = \chi_{16}(d) f(z). \tag{3.7}$$

Note that we still need to consider the cases when $c = 0$ and $c < 0$. If $c = 0$, then $a = d = \pm 1$. Therefore, (3.7) still holds.

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $c < 0$, then by (2.1) and (3.7) we have

$$\begin{aligned} \left(f(z) \mid \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_3\right) \mid [\gamma]_3 &= f(z) \mid \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}_3 \\ &= \chi_{16}(-d) f(z) \\ &= -\chi_{16}(d) f(z). \end{aligned}$$

While on the other hand, we also have

$$f(z) \mid \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_3 = -f(z).$$

Therefore, we proved that $f(z) \mid [\gamma]_3 = \chi_{16}(d) f(z)$ holds for all matrices $\gamma \in \Gamma_0(16)$.

Next we have to check conditions in (2.3) and (2.4). Let $g(z) = \eta^6(z)$. So we have

$$f(z) = g(4z) = \frac{1}{4^{3/2}} \cdot g(z) \mid \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}_3 .$$

Then for $\gamma_0 = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where as above we may assume $c_0 > 0$, the identity (2.1) gives

$$\begin{aligned} f(z) \mid [\gamma_0]_3 &= \frac{1}{8} \left(g(z) \mid \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}_3 \right) \mid [\gamma_0]_3 \\ &= \frac{1}{8} \cdot g(z) \mid \begin{pmatrix} 4a_0 & 4b_0 \\ c_0 & d_0 \end{pmatrix}_3 . \end{aligned}$$

To find the Fourier expansion of $f(z) \mid [\gamma_0]_3$, we first prove that there exist positive integers A, D and an integer B, such that

$$\gamma := \begin{pmatrix} 4a_0 & 4b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} \in \mathrm{SL}_2(\mathbb{Z}). \tag{3.8}$$

By multiplying out, we get

$$\gamma = \begin{pmatrix} 4a_0/A & (4b_0A - 4a_0B)/AD \\ c_0/A & (d_0A - c_0B)/AD \end{pmatrix}.$$

Let $A = \gcd(c_0, 4)$ and $D = 4/A$, then $A, D \in \mathbb{Z}^+$ and $AD = 4$, so

$$\gamma = \begin{pmatrix} a_0D & b_0A - a_0B \\ c_0/A & (d_0A - c_0B)/4 \end{pmatrix},$$

and $\det \gamma = 1$. Also it is clear that c_0/A, the lower-left entry of γ, is an integer. Now we show that the there exists B such that the lower-right entry is also an integer. Since $A = \gcd(c_0, 4)$, we have $\gcd(c_0/A, 4/A) = 1$. Thus there is a solution to the congruence $(c_0/A)\, x \equiv d_0 \pmod{4/A}$. So there exists $B \in \mathbb{Z}$ such that $(d_0A - c_0B)/4 \in \mathbb{Z}$. This shows that matrices as in (3.8) exist. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Note that since $c_0 > 0$, we have $c > 0$. Then we have

$$
\begin{aligned}
f(z) \mid [\gamma_0]_3 &= \frac{1}{8} \cdot g(z) \mid \begin{pmatrix} 4a_0 & 4b_0 \\ c_0 & d_0 \end{pmatrix}_3 \\
&= \frac{1}{8} \left(g \mid [\gamma]_3 \right) \mid \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}_3 \\
&= \frac{1}{8} \left(\eta^6(\gamma z)(cz+d)^{-3} \right) \mid \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}_3 \\
&= \frac{1}{8} \varepsilon(a,b,c,d)^6 \left(\eta^6(z) \mid \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}_3 \right) \\
&= \frac{1}{8} \varepsilon(a,b,c,d)^6 \left(4^{3/2} D^{-3} \eta^6 \left(\frac{Az+B}{D} \right) \right) \\
&= M \cdot \eta^6 \left(\frac{Az+B}{D} \right) \quad \text{where } M = \frac{1}{8} \varepsilon(a,b,c,d)^6 4^{3/2} D^{-3} \\
&= M \cdot \left(e^{2\pi i \frac{Az+B}{D}} \right)^{1/4} \prod_{n=1}^{\infty} (1 - \left(e^{2\pi i \frac{Az+B}{D}} \right)^n)^6 \\
&= e^{\pi i B/2D} M \cdot q_{4D}^{A} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i B n/D} q_{4D}^{4An} \right)^6, \quad \text{where } q_D = e^{2\pi i z/D}.
\end{aligned}
$$

It is now clear that the Fourier expansion of $f(z) \mid [\gamma_0]_3$ has $a(n) = 0$ for $n \leq 0$. Therefore, conditions in (2.2), (2.3) and (2.4) are satisfied, and $f(z) = \eta^6(4z)$ is a cusp-form of weight 3 on $\Gamma_0(16)$ with the character χ_{16}. In other words,

$$f(z) \in S_3(\Gamma_0(16), \chi_{16}).$$

The same techniques can be applied to $\eta^{12}(2z), \eta^4(6z)$ and $\eta^2(12z)$, and the other three statements in Lemma 2 can be proved similarly. $\square$

Proof of Lemma 3. Since the four functions are all cusp-forms, we thus can use the following formula [C-O] to compute the dimension of their spaces. Let $\dim M_k(\Gamma_0(N), \chi)$ and $\dim S_k(\Gamma_0(N), \chi)$ denote the dimension of the space of corresponding modular forms and cusp-forms, respectively. Then we have

$$
\begin{aligned}
&\dim S_k \left(\Gamma_0(N), \chi \right) - \dim M_{2-k} \left(\Gamma_0(N), \chi \right) \\
&= \frac{k-1}{12} N \prod_{p \mid N} \left(1 + \frac{1}{p} \right) - \frac{1}{2} \prod_{p \mid N} \lambda(r_p, s_p, p) + \varepsilon_k \sum_{\substack{x \mod N \\ x^2+1 \equiv 0 \mod N}} \chi(x) + \mu_k \sum_{\substack{x \mod N \\ x^2+x+1 \equiv 0 \mod N}} \chi(x),
\end{aligned}
$$

where r_p (resp. s_p) denotes the exponent of p in the factorization of N (resp. of the conductor of the character χ), and $\lambda(r_p, s_p, p)$, ε and μ are defined as

$$\lambda(r_p, s_p, p) := \begin{cases} p^{r'} + p^{r'-1} & \text{if } 2s_p \le r_p = 2r', \\ 2p^{r'} & \text{if } 2s_p \le r_p = 2r'+1, \\ 2p^{r_p - s_p} & \text{if } \ 2s_p > r_p, \end{cases}$$

$$\varepsilon_k := \begin{cases} 0 & \text{if } k \text{ is odd}, \\ -\frac{1}{4} & \text{if } k \equiv 2 \pmod 4, \\ \frac{1}{4} & \text{if } k \equiv 0 \pmod 4, \end{cases} \qquad \mu_k := \begin{cases} 0 & \text{if } k \equiv 1 \pmod 3, \\ -\frac{1}{3} & \text{if } k \equiv 2 \pmod 3, \\ \frac{1}{3} & \text{if } k \equiv 0 \pmod 3. \end{cases}$$

Note that

$$\dim M_k(\Gamma_0(N), \chi) = 0 \quad \text{if } k < 0 \text{ or } k = 0 \text{ and } \chi \text{ is not the trivial character } \chi^{\text{triv}};$$
$$\dim M_0(\Gamma_0(N), \chi^{\text{triv}}) = 1.$$

And we also know that the conductor of the trivial character is 1, and the conductor of χ_N, where $4 \mid N$, is 4.

Since $\eta^6(4z) \in S_3(\Gamma_0(16), \chi_{16})$, we have $\dim M_{-1}(\Gamma_0(16), \chi_{16}) = 0$, $r_2 = 4$, $s_2 = 2$, $\varepsilon_3 = 0$ and $\mu_3 = \frac{1}{3}$. We get by the formula above

$$\dim S_3(\Gamma_0(16), \chi_{16}) = (1/6) \cdot 16 \cdot (3/2) - (1/2)\left(2^2 + 2\right) = 1.$$

Similarly, we can find that $\dim S_2(\Gamma_0(36), \chi_{36}^{\text{triv}}) = 1$ and $\dim S_6(\Gamma_0(4), \chi_4^{\text{triv}}) = 1$. In addition, by (2.5), there must exist a scalar λ_p for every prime p such that $f(z) \mid T(p) = \lambda_p f(z)$, i.e. $\eta^{12}(2z)$, $\eta^6(4z)$ and $\eta^4(6z)$ are all primitive eigenforms. By [S], we know that $\eta^2(12z)$ is also a primitive eigenform. $\square$

Since Theorem 2 immediately follows from Theorem 3, we will conclude our argument by proving Theorem 3.

Proof of Theorem 3. Let $f(z) = \eta^6(4z)$. Since it is a cusp-form of weight 3 on $\Gamma_0(16)$ with the character χ_{16}, by lemma 3 it is a primitive eigenform. We know by (2.6) that for each prime l, there exists a scalar λ_l such that

$$f(z) \mid T(l) = \sum_{n=1}^{\infty} \left(b(ln) + \chi_{16}(l) l^2 b(n/l)\right) q^n = \lambda_l f(z) = \sum_{n=1}^{\infty} \lambda_l b(n) q^n.$$

It is clear that for all non-negative integers n and primes l, we have

$$\lambda_l b(n) = b(ln) + \chi_{16}(l) l^2 b(n/l).$$

Since $b(1) = 1$, we get $\lambda_l = b(l)$. Noticing that $\chi_{16}(l) = \left(\frac{-4}{l}\right)$ for all primes l, we have

$$b(l)b(n) = b(ln) + \left(\frac{-4}{l}\right) l^2 b(n/l),$$

for all primes l and positive integers n. Thus we obtain the desired identity (3.2). By applying the same technique, (3.1), (3.3) and (3.4) can be proved in a similar way. $\square$

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