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The k-Compartment Problem

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Abstract

This article defines a new minimization problem, the *k-Compartment Problem*, and presents its solution. The k-Compartment Problem is to determine the minimum sum of $k+1$ line segments that intersect two parallel lines which form k compartments.

1 Introduction

With the advent of geometric measure theory in the 1960's, a new realm of mathematics was born and with it the ability to solve advanced geometric problems [3]. Geometric measure theorists often attempt to minimize the measure of some quantity associated with a class of geometric objects that satisfy some specified criteria. This article presents the solution of a new minimization problem named the *k-Compartment Problem*, which determines the minimum sum of $k+1$ line segments that intersect two parallel lines. Before, we begin our discussion of the k-Compartment Problem we introduce two examples of minimization problems: the Steiner Problem and the Planar Soap Bubble Problem.

The Steiner problem arose in the early 19^{th} century, when Jacob Steiner wanted to know the minimum road length to connect three villages: A, B, and C [1]. He assumed that the villages were in the plane and proved two cases. First, if the largest angle in the triangle ABC is larger than 120^0 (suppose the vertex is located at C) then the shortest road is constructed by joining line segments AC and BC. Second, if the largest angle in the triangle ABC is less than 120^0, then a fourth point P lies in ABC with AP, BP, and CP meeting at 120^0 angles. These three line segments form the minimum road that connects the three villages. A generalized Steiner problem aims at minimizing a "street network" that connects n points in the Euclidean plane. Much is known about the generalized Euclidean Steiner Problem, but what if the Steiner Problem were extended to minimize n points on any surface? For example, consider the Steiner Problem in hyperbolic space. Suddenly, a new set of problems arises that have yet to be solved.

Similarly, the Planar Soap Bubble Problem is to find the least perimeter way to enclose and separate n given areas on the plane. Recently, Wichiramala solved the Triple Planar Soap Bubble Problem for his doctoral thesis at the University of Illinois [5]. He proved that the standard triple bubble uniquely encloses three given areas in the plane with the least total perimeter. However, the Planar Soap Bubble Problem has yet to be solved.

We introduced the Steiner and Planar Soap Bubble Problems because we believe the k-Compartment Problem will aid in their solutions. For example, Dimond used our result in her proof of the minimum network needed to connect three equidistant points in hyperbolic space [2]. This article is divided into four parts: first, definition of the 1-Compartment Problem; second, introduction of new notation and properties of flux through a line segment; third, definition of the k-Compartment Problem and the minimizing proof; and fourth, implications of the result.

2 Problem

In order to better understand the k-Compartment Problem, first we define the 1-Compartment Problem.

Definition 2.1. In figure 1, α and β are parallel lines a distance h apart, and edges a_1 and b_1 have fixed lengths. The vertices of a_1 and b_1 form a quadrilateral or *compartment* with additional edges c_1 and c_2. The *1-Compartment Problem* is to determine the minimum of the sum of the lengths of c_1 and c_2 as a_1 is allowed to move horizontally along α while b_1 remains stationary.

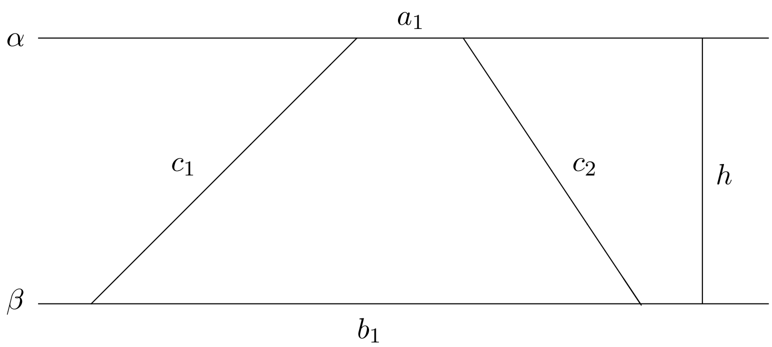

Figure 1: The 1-Compartment Problem

Note that the area of the quadrilateral formed by edges a_1, b_1, c_1, and c_2 remains the same regardless of the location of a_1 along α. All configurations that satisfy these constraints will be referred to as competitors (see figure 2). Hence, the goal is to determine the minimum length of the sum c_1 and c_2 among all competitors.

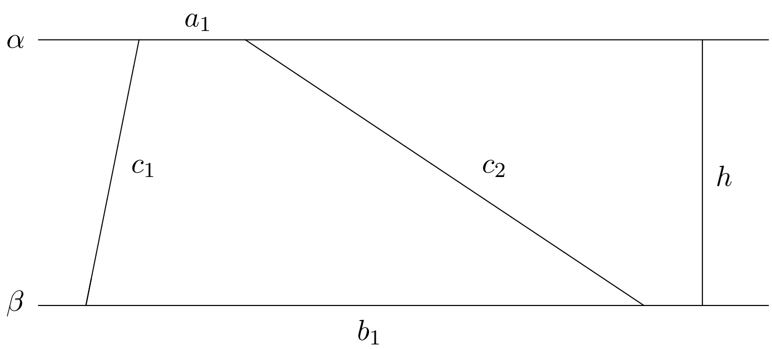

Figure 2: Notice how a_1 has shifted to the left, but b_1 has remained in the same position. This is a competing configuration.

3 Preliminaries

Before the necessary background for the proof is discussed, we will introduce two new notations. Henceforth, the length of a line segment a_i will be denoted as $l(a_i)$, and the flux through a line segment a_i by the vector field $\vec{v}_i$ will be denoted as $F(\vec{v}_i, a_i)$. We note that the idea of using flux arguments with multiple vector fields for minimization problems is not new and its origins can be found in a paper written by Lawlor and Morgan [4].

Now, a review of the divergence theorem will be useful in understanding this proof. The most familiar form of the divergence theorem is in $\mathbb{R}^3$ and is stated as follows:

> Let T be a volume that is bounded by a simple, closed, piecewise-smooth orientable surface S. Let $\vec{v}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains T. Then
>
> $$\int\int\int_T \operatorname{div} \vec{v}\, dV = \int\int_S \vec{v} \cdot \vec{n}\, d\sigma$$
>
> Where $\vec{n}$ is a normal vector to surface S and σ is the surface area of S.

For our purposes we will need a form of the divergence theorem in $\mathbb{R}^2$, the normal form of Green's theorem, which is stated as follows:

> Let R be a piecewise-smooth region with a piecewise-smooth orientable boundary C. Let $\vec{v}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^2$ that contains R. Then
>
> $$\int\int_R \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) dA = \oint_C \vec{v} \cdot \vec{n}\, ds$$
>
> Where $\vec{n}$ is the a normal vector to C and s is the arclength of C.

The following facts, which are easily verified, provide the basis necessary to prove the k-Compartment Problem. Let C be a closed loop, let $\vec{v}$ be a constant vector field, and let σ be a line segment and $\vec{n}$ a unit normal vector on segment σ. Then

1. The flux $\oint_C \vec{v} \cdot \vec{n}\, ds$ through C will be zero, which follows from the normal form of Green's theorem.

2. If $\vec{v}$ is parallel to and in the direction of $\vec{n}$ the value of the flux integral will be $l(\sigma)$.

3. If $\vec{v}$ is not parallel to and not in the direction of $\vec{n}$, the value of the flux integral will be less than $l(\sigma)$.

   A fourth observation is based on the results above. Let $\vec{v}$ be a horizontal unit vector field in a two-dimensional pipe, and let a and b denote two line segments that intersect the pipe such that their end points lie on the pipe boundary (see figure 3). Also, let $\vec{n_a}$ and $\vec{n_b}$ be unit normals to line segments a and b respectively.

4. As a result of (2) and (3) if $\vec{v}$ is in the direction of $\vec{n}_a$ but not in the direction of $\vec{n}_b$, then the following is true:

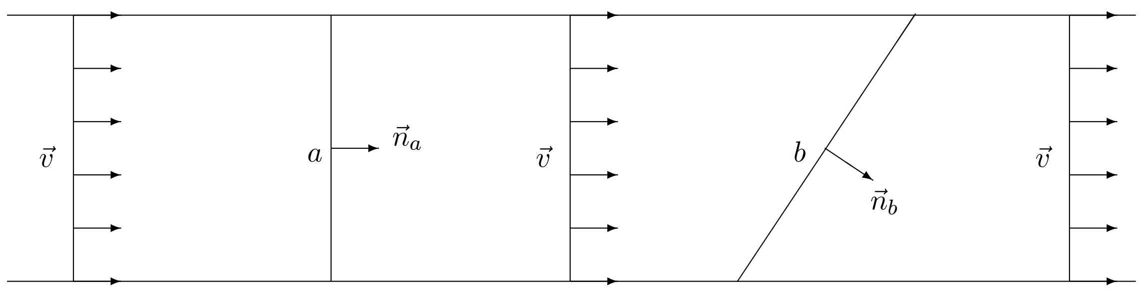

Figure 3: A two-dimensional pipe with constant vector field $\vec{v}$.

(a) The flux through a and b is the same.

$$F(\vec{v}, a) = \int_a \vec{v} \cdot \vec{n}_a \, ds = \int_b \vec{v} \cdot \vec{n}_b \, ds = F(\vec{v}, b)$$

Remark: as previously defined in the preliminaries, $F(\vec{v}_i, a_i)$ denotes the flux through through a line segment a_i by the vector $\vec{v}_i$

(b) The value of the flux integral will be $l(a)$ but only a fraction of $l(b)$

$$F(\vec{v}, a) = \int_a \vec{v} \cdot \vec{n}_a \, ds = l(a)$$

$$F(\vec{v}, b) = \int_b \vec{v} \cdot \vec{n}_b \, ds \leq l(b)$$

(c) Therefore, because of (4a) $l(a)$ will be less than or equal to $l(b)$.

$$l(a) \leq l(b)$$

This inequality is essential to proving the minimizing case in the k-Compartment Problem. It is the disparity in lengths that will allow one to differentiate between the minimizing case and a competitor.

4 Technique

Now that we have reviewed flux and its importance in this proof, we define the k-Compartment Problem.

Definition 4.1. In figure 4, α and β are two parallel lines a distance h apart. The edges $a_1, a_2, a_3, \ldots, a_k$ and $b_1, b_2, b_3, \ldots, b_k$ are placed end to end and have fixed lengths. The vertices of $a_1, a_2, a_3, \ldots, a_k$ and $b_1, b_2, b_3, \ldots, b_k$ form k quadrilaterals or *compartments* with additional edges $c_1, c_2, c_3, \ldots, c_k, c_{k+1}$. The *k-Compartment Problem* is to determine the minimum of the sum of the lengths of $c_1, c_2, c_3, \ldots, c_k, c_{k+1}$ as $a_1, a_2, a_3, \ldots, a_k$ are allowed to move together horizontally along α while $b_1, b_2, b_3, \ldots, b_k$ remain stationary.

In preparation for the following theorem, we define the incline angle for c_i in figure 4 as the angle θ_i formed between c_i and b_i. Note that as $a_1, a_2, \ldots, a_k$ move together along α that the incline angle θ_i will change. We also define $A = a_1 \cup a_2 \cup \ldots \cup a_k$, and further note that the configurations that satisfy definition 4.1 can be parameterized in t where t represents the position of A along α.

Now, we consider a parameterized function that describes the movement of A along α . Let

$$f(\theta_1, \theta_2, \ldots, \theta_{k+1}) = \cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_{k+1}$$

and

$$\vec{r}(t) = \langle \theta_1(t), \theta_2(t), \ldots, \theta_{k+1}(t) \rangle$$

First, we consider the bounds of f. If a shifts in the negative direction (to the left) a large amount, then the incline angles $\theta_1, \theta_2, \ldots, \theta_{k+1}$ approach π. Hence the value of f in this configuration would approach $-(k+1)$. On the other hand if a shifts in the positive direction (to the right) a large amount, then the incline angles $\theta_1, \theta_2, \ldots, \theta_{k+1}$ approach 0. Hence the value of f is this configuration would approach $(k+1)$. Therefore, f is bounded below by $-(k+1)$ and above by $(k+1)$.

Second, we consider the behavior of f between its bounds, so we take the derivative of f with respect to t.

$$\begin{aligned} f'(\theta_1(t), \theta_2(t), \ldots, \theta_{k+1}(t)) &= f'(\vec{r}(t)) \\ &= \nabla f \cdot r'(t) \\ &= -\theta_1'(t)\sin[\theta_1(t)] - \theta_2'(t)\sin[\theta_2(t)] - \ldots - \theta_{k+1}'(t)\sin[\theta_{k+1}(t)] \end{aligned}$$

We consider the case where a moves in the positive direction. Since a is moving to the right $\theta_i(t)$ is decreasing and thus $\theta_i'(t)$ is negative. We also know that $\sin[\theta_i(t)]$ is always positive between 0 and π (the lower and upper limits for θ_i). Therefore,

$$f'(\theta_1(t), \theta_2(t), \ldots, \theta_{k+1}(t)) > 0$$

Since $f'(\theta_1(t), \theta_2(t), \ldots, \theta_{k+1}(t))$ is always positive, f is always increasing. In addition, if f is always increasing and known to take on a positive and negative value, then f has exactly one zero. We will now show, in the following theorem, that the configuration that is a solution to the k-Compartment Problem is the configuration that satisfies

$$f(\theta_1, \theta_2, \ldots, \theta_{k+1}) = \cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_k + \cos\theta_{k+1} = 0$$

Theorem 4.2 (The k-Compartment Problem Theorem)**.** *Given a k-Compartment Problem as described in definition 4.1, the solution is the configuration with angles θ_i such that*

$$f(\theta_1, \theta_2, \ldots, \theta_{k+1}) = \cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_k + \cos\theta_{k+1} = 0$$

Proof. Let $\vec{m}_i$, $\vec{n}_i$, and $\vec{u}_i$ be unit normal vectors to a_i, b_i, and c_i respectively. Note that $\vec{m}_i = \langle 0, 1 \rangle$ and $\vec{n}_i = \langle 0, -1 \rangle$ for all i. Let $\vec{v}_i$ be the constant vector field that assigns each point in the plane the vector $\vec{u}_i$. In what follows, we use the non-starred notation to indicate the minimizing configuration and the $*$ notation to indicate objects of a competing configuration.

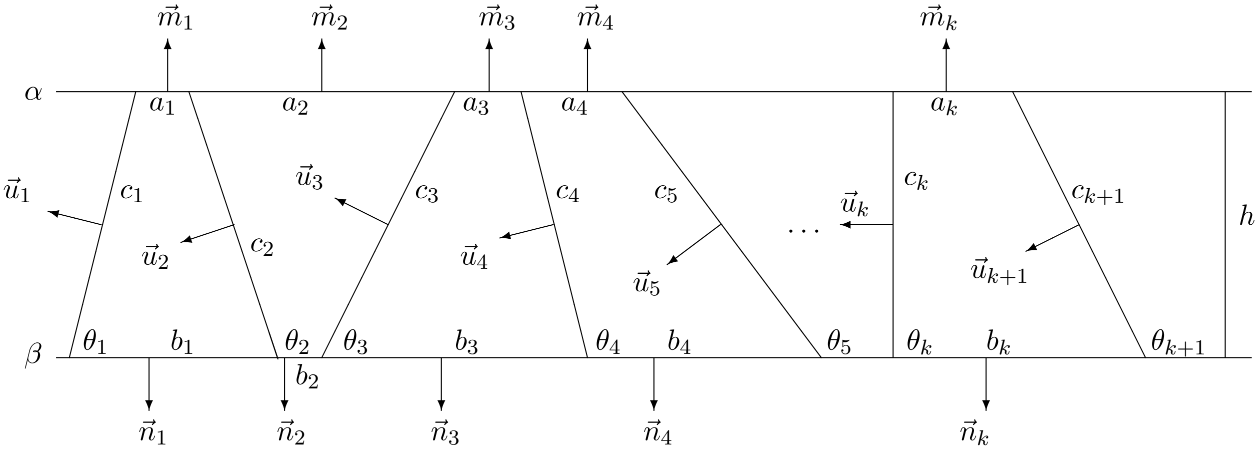

Figure 4: The k-Compartment Problem.

The proof is organized into three steps.

Step 1. Show

$$F(\vec{v}_1, c_1) + F(\vec{v}_2, c_2) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) = l(c_1) + l(c_2) + \ldots + l(c_{k+1})$$

Step 2. Show

$$F(\vec{v}_1, c_1^*) + F(\vec{v}_2, c_2^*) + \ldots + F(\vec{v}_{k+1}, c_{k+1}^*) \leq l(c_1^*) + l(c_2^*) + \ldots + l(c_{k+1}^*)$$

Step 3. Show

$$F(\vec{v}_1, c_1) + F(\vec{v}_2, c_2) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) = F(\vec{v}_1, c_1^*) + F(\vec{v}_2, c_2^*) + \ldots + F(\vec{v}_{k+1}, c_{k+1}^*)$$

From steps 1, 2, and 3 it follows that:

$$l(c_1) + l(c_2) + \ldots + l(c_{k+1}) \leq l(c_1^*) + l(c_2^*) + \ldots + l(c_{k+1}^*)$$

From the preliminary discussion, it is easy to show step 1. Since $\vec{v}_i$ is normal to each line segment c_i, the flux through c_i will be the its length $l(c_i)$ (see 4b). Therefore, the sum of the flux values will equal the sum of the lengths of c_i.

Step 2 follows a similar argument. Since $\vec{v}_i$ is not necessarily normal to each line c_i^*, the flux through c_i^* will be less than or equal to its length $l(c_i^*)$ (see 4b). Therefore, the sum of the flux values will be less than or equal the sum of the lengths of c_i^*.

Before we began to address step 3, it is necessary to establish several identities.

First, because each compartment is closed we note the following identities by the discussion in statement (1) of the preliminary section.

$$
\begin{aligned}
F(\vec{v}_1, c_1) + F(\vec{v}_1, a_1 + a_2 + \ldots + a_k) - F(\vec{v}_1, c_{k+1}) + F(\vec{v}_1, b_1 + b_2 + \ldots + b_k) &= 0 \quad (1)\\
F(\vec{v}_1, c_1^*) + F(\vec{v}_1, a_1^* + a_2^* + \ldots + a_k^*) - F(\vec{v}_1, c_{k+1}^*) + F(\vec{v}_1, b_1^* + b_2^* + \ldots + b_k^*) &= 0 \quad (2)\\
F(\vec{v}_2, c_2) + F(\vec{v}_2, a_2 + a_3 + \ldots + a_k) - F(\vec{v}_2, c_{k+1}) + F(\vec{v}_2, b_2 + b_3 + \ldots + b_k) &= 0 \quad (3)\\
F(\vec{v}_2, c_2^*) + F(\vec{v}_2, a_2^* + a_3^* + \ldots + a_k^*) - F(\vec{v}_2, c_{k+1}^*) + F(\vec{v}_2, b_2^* + b_3^* + \ldots + b_k^*) &= 0 \quad (4)\\
&\vdots\\
F(\vec{v}_k, c_k) + F(\vec{v}_k, a_k) - F(\vec{v}_k, c_{k+1}) + F(\vec{v}_k, b_k) &= 0 \quad (5)\\
F(\vec{v}_k, c_k^*) + F(\vec{v}_k, a_k^*) - F(\vec{v}_k, c_{k+1}^*) + F(\vec{v}_k, b_k^*) &= 0 \quad (6)
\end{aligned}
$$

Note that the third term in equations (1)-(6) is preceded by a negative. This is due to the statement of the divergence theorem and the notion of orientability. An outward normal indicates a positive while an inward normal indicates a negative. Hence, because the third unit normal of each compartment is inward, the third term is preceded by a negative.

Figures 5, 6, and 7 indicate the closed loops from which equations (1), (3), and (5) are derived. Notice that the equations are not derived from each individual compartment side by side, but rather one large compartment that shrinks k times.

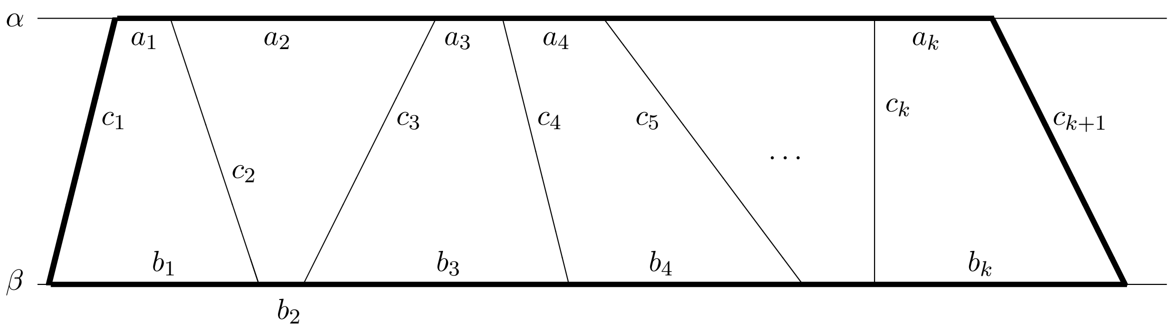

Figure 5: The compartment shown by equation (1).

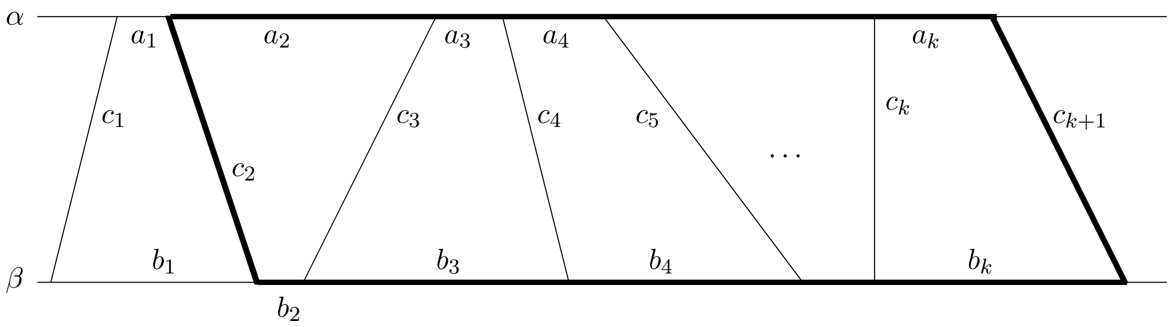

Figure 6: The compartment shown by equation (3).

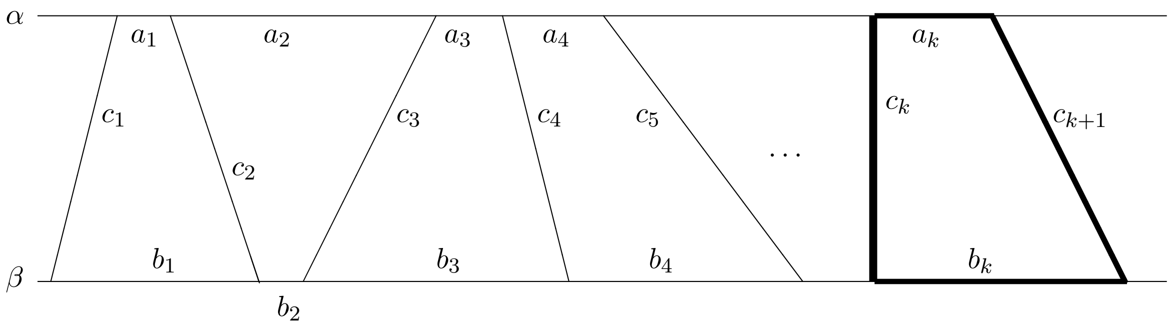

Figure 7: The compartment shown by equation (5).

Now, from the problem definition the flux through the fixed line segments, a_i and b_i will be identical for the minimizer and any competiting case because the line segments are of equal length and in the same position for both cases.

$$\begin{aligned}
F(\vec{v}_1, a_1 + a_2 + \ldots + a_k) &= F(\vec{v}_1, a_1^* + a_2^* + \ldots + a_k^*) && (7)\\
F(\vec{v}_2, a_2 + a_3 + \ldots + a_k) &= F(\vec{v}_2, a_2^* + a_3^* + \ldots + a_k^*) && (8)\\
&\vdots \\
F(\vec{v}_k, a_k) &= F(\vec{v}_k, a_k^*) && (9)\\
& && (10)\\
F(\vec{v}_1, b_1 + b_2 + \ldots + b_k) &= F(\vec{v}_1, b_1^* + b_2^* + \ldots + b_k^*) && (11)\\
F(\vec{v}_2, b_2 + b_3 + \ldots + b_k) &= F(\vec{v}_2, b_2^* + b_3^* + \ldots + b_k^*) && (12)\\
&\vdots \\
F(\vec{v}_k, b_k) &= F(\vec{v}_k, b_k^*) && (13)
\end{aligned}$$

Before we continue with the proof, there is one more identity that needs to be established. In order to do so, we prove the following lemma.

Lemma 4.2.1. *The sum of vector fields $\vec{v}_1$ through $\vec{v}_{k+1}$ only has a horizontal component.*

Proof. Figure 8 shows a detail of the k-Compartment Problem. Let $\vec{u}_i = \langle -\gamma_i, \delta_i \rangle$. Then $\cos\theta_i = \delta_i$ since $\vec{u}_i$ is unit length and δ_i represents the component of the vector field $\vec{v}_i$ in the vertical direction. In order for the sum of vector fields $\vec{v}_1$ through $\vec{v}_{k+1}$ to only have a horizontal component, the sum of the vertical components must be zero.

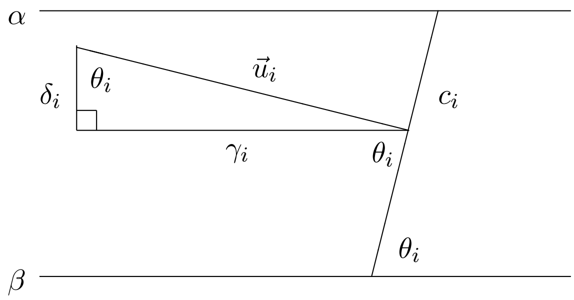

Figure 8: A graphical representation of the importance of the $\cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_k + \cos\theta_{k+1} = 0$ identity.

Since $\cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_{k+1} = 0$, then

$$\delta_1 + \delta_2 + \ldots + \delta_{k+1} = 0$$

Therefore, the sum of the vector fields $\vec{v}_i$ through $\vec{v}_{k+1}$ yields only a horizontal component.

$\square$

As a result of Lemma 4.2.1 the flux through c_{k+1} and c^*_{k+1} will be equal as noted in (4a) of the preliminaries discussion.

$$F(\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k, c_{k+1}) = F(\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k, c^*_{k+1}) \tag{14}$$

Now that we have established the identities that are used in the proof, we show step 3:

$$F(\vec{v}_1, c_1) + F(\vec{v}_2, c_2) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) = F(\vec{v}_1, c^*_1) + F(\vec{v}_2, c^*_2) + \ldots + F(\vec{v}_{k+1}, c^*_{k+1})$$

First, we will work with the left side of the above equation. Solve identity equations (1), (3), and (5) for $F(\vec{v}_1, c_1)$, $F(\vec{v}_2, c_2)$, and $F(\vec{v}_k, c_k)$ respectively and substitute.

$$\begin{aligned}
F(\vec{v}_1, c_1) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) \;=\;& -F(\vec{v}_1, a_1 + a_2 + \ldots + a_k) + F(\vec{v}_1, c_{k+1}) - F(\vec{v}_1, b_1 + b_2 + \ldots + b_k) \\
& -F(\vec{v}_2, a_2 + a_3 + \ldots + a_k) + F(\vec{v}_2, c_{k+1}) - F(\vec{v}_2, b_2 + b_3 + \ldots + b_k) \\
& \vdots \\
& -F(\vec{v}_k, a_k) + F(\vec{v}_k, c_{k+1}) - F(\vec{v}_k, b_k) \\
& +F(\vec{v}_{k+1}, c_{k+1})
\end{aligned}$$

Simplify the above expression using identity equations (7) through (12). Also note that terms $F(\vec{v}_1, c_{k+1})$, $F(\vec{v}_2, c_{k+1})$, $F(\vec{v}_k, c_k)$, and $F(\vec{v}_{k+1}, c_{k+1})$ can be combined into one expression, $F(\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_{k+1}, c_{k+1})$, which can be replaced with $F(\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_{k+1}, c^*_{k+1})$ because of identity (13).

$$\begin{aligned}F(\vec{v}_1, c_1) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) \quad = \quad & -F(\vec{v}_1, a_1^* + a_2^* + \ldots + a_k^*) - F(\vec{v}_1, b_1^* + b_2^* + \ldots + b_k^*) \\ & -F(\vec{v}_2, a_2^* + a_3^* + \ldots + a_k^*) - F(\vec{v}_2, b_2^* + b_3^* + \ldots + b_k^*) \\ & \vdots \\ & -F(\vec{v}_k, a_k^*) - F(\vec{v}_k, b_k^*) \\ & +F(\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_{k+1}, c_{k+1}^*)\end{aligned}$$

Split $F(\vec{v}_1+\vec{v}_2+\ldots+\vec{v}_{k+1}, c_{k+1}^*)$ into individual components $F(\vec{v}_1, c_{k+1}^*)$, $F(\vec{v}_2, c_{k+1}^*)$, $F(\vec{v}_k, c_{k+1}^*)$ and $F(\vec{v}_{k+1}, c_{k+1}^*)$ and reorganize the expression.

$$\begin{aligned}F(\vec{v}_1, c_1) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) \quad = \quad & -F(\vec{v}_1, a_1^* + a_2^* + \ldots + a_k^*) + F(\vec{v}_1, c_{k+1}^*) - F(\vec{v}_1, b_1^* + b_2^* + \ldots + b_k^*) \\ & -F(\vec{v}_2, a_2^* + a_3^* + \ldots + a_k^*) + F(\vec{v}_2, c_{k+1}^*) - F(\vec{v}_2, b_2^* + b_3^* + \ldots + b_k^*) \\ & \vdots \\ & -F(\vec{v}_k, a_k^*) + F(\vec{v}_k, c_{k+1}^*) - F(\vec{v}_k, b_k^*) \\ & +F(\vec{v}_{k+1}, c_{k+1}^*)\end{aligned}$$

Using identities (2), (4), and (6) this expression can be simplified to:

$$F(\vec{v}_1, c_1) + \ldots + F(\vec{v}_{k+1}, c_{k+1}) = F(\vec{v}_1, c_1^*) + F(\vec{v}_2, c_2^*) + \ldots + F(\vec{v}_{k+1}, c_{k+1}^*)$$

Which, because of steps 1, 2, and 3 is equivalent to:

$$l(c_1) + l(c_2) + \ldots + l(c_{k+1}) \leq l(c_1^*) + l(c_2^*) + \ldots + l(c_{k+1}^*)$$

Therefore, the minimizing configuration of the *k-Compartment Problem* is the configuration that satisfies $\cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_k + \cos\theta_{k+1} = 0$. In addition, as discussed in the paragraphs immediately preceding this theorem, the minimizing configuration is unique.

□

5 Conclusion

In summary, we have reviewed the normal form of Green's theorem and formed a flux argument. Then, we defined the k-Compartment Problem and the conjectured minimizer. Finally, in three steps we showed that the minimizing configuration is unique and satisfies

$$f(\theta_1, \theta_2, \ldots \theta_{k+1}) = \cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_k + \cos\theta_{k+1} = 0$$

Now that we have proved the minimizing configuration, what is the significance of this result? The results of this proof are a part of a new technique called mapped slicing. For example, mapped slicing is used to prove the minimum path to connect three equidistant points in hyperbolic space [2]. In her paper, Diamond needed the solution to the 1-compartment problem to show the minimum distance needed to connect three equidistant points in hyperbolic space. It is hoped that a solution to the k-Compartment Problem will aid in the solution of Steiner's Problem of n points in hyperbolic space.

In addition, it is also hoped that mapped slicing, of which the solution to the k-Compartment Problem is an integral element, will aid in the solution of the Planar Soap Bubble Problem.

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