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Exceptional Automorphisms of (Generalized) Super Elliptic Surfaces

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Dedicated to Emilio Bujalance for his sixtieth anniversary

Abstract. A *super-elliptic* surface is a compact, smooth Riemann surface S with a conformal automorphism w of prime order p such that $S/\langle w\rangle$ has genus zero, extending the hyper-elliptic case $p=2$. More generally, a cyclic n-gonal surface S has an automorphism w of order n such that $S/\langle w\rangle$ has genus zero. All cyclic $n-$gonal surfaces have tractable defining equations. Let $A=\mathrm{Aut}(S)$ and N be the normalizer of $C=\langle w\rangle$ in A. The structure of N, in principal, can be easily determined from the defining equation. If the genus of S is sufficiently large in comparison to n, and C satisfies a generalized super-elliptic condition, then $A=N$. For small genus $A-N$ may be non-empty and, in this case, any automorphism $h\in A-N$ is called *exceptional*. The exceptional automorphisms of super-elliptic surfaces are known whereas the determination of exceptional automorphisms of all general cyclic n-gonal surfaces seems to be hard. We focus on *generalized super-elliptic* surfaces in which n is composite and the projection of S onto S/C is fully ramified. Generalized super-elliptic surfaces are easily identified by their defining equations. In this paper we discuss an approach to the determination of generalized super-elliptic surfaces with exceptional automorphisms.

1. Cyclic n-gonal surfaces

1.1. Cyclic n-gonal surfaces - introduction. A cyclic n-gonal surface (curve) is a compact, smooth Riemann surface with a plane model of the form

$$y^n = f(x) = \prod_{i=1}^{s}(x-a_i)^{t_i} \tag{1.1}$$

where a_i, t_i and $t=t_1+\cdots+t_s=deg(f)$ satisfy

- the a_i are distinct,
- $0<t_i<n$,
- n divides t (this is not the typical requirement), and

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- $gcd(n, t_1, \ldots, t_s) = 1$.

If $n = 2$ then the surface is hyperelliptic.

The plane model of the surface is smooth except at the points $(a_i, 0)$ where $t_i > 1$, and at a single point at ∞ if $t > n$. Recall that the normalization $S^\nu \to S$ desingularizes S by pulling apart local branches and smoothing out the resulting cusps (if any). The map is an isomorphism away from the singular points. At a singular point of $z \in S$ the local branches are the components of $U - \{z\}$ where U is a small connected neighbourhood of z in S. Thus, for such a z, $\nu^{-1}(z)$ will consist of d_z separate points. Alternatively, we may think of constructing S^ν by removing the singular points of S and them smoothly completing the resulting punctured surface. By considering a local analysis of the defining equation at the singular points, it is easily shown that the normalization $S^\nu \to S$ has $d_i = gcd(t_i, n)$ points lying over $(a_i, 0)$ and $n = gcd(t, n)$ points lying over ∞. We call S^ν the *smooth model* and S the *plane model* though we frequently loosely identify the two surfaces. The genus σ of S^ν is given by

$$\sigma = \frac{1}{2}\left(2 + (s-2)n - \sum_{i=1}^{s} d_i\right). \tag{1.2}$$

If ω is a n^{th} root of unity, then $(x, y) \to (x, \omega y)$ is an automorphism of S which fixes the points $(a_i, 0)$ and no others. Let C be the cyclic group of automorphisms obtained by letting ω range over all n^{th} roots of unity. The action of C on S, and its lift to S^ν, is called a *cyclic n-gonal action*. The map $\pi : S^\nu \to S \to \mathbb{P}^1$, $(x, y) \to x$ is a quotient map for the projection $S^\nu \to S^\nu/C$, and is called the *cyclic n-gonal morphism*. The degree of ramification of π over a_i is $n_i = n/gcd(t_i, n)$. In fact there are $d_i = gcd(t_i, n)$ points lying over a_i, and at each such point P the stabilizer of the C action, C_P, is the unique subgroup of C of order n_i. The quotient group C/C_P transitively permutes the points lying over $\pi(P)$. The map is unramified over ∞ because n divides t and there are n distinct points over ∞. Let w be the generator of C corresponding to $\omega = \exp(2\pi i/n)$. At any point in S^ν lying over lying over a_j, the rotation number of w is $\exp\left(\frac{2\pi i t_j}{n}\right)$.

For the hyperelliptic case, $n = 2$, $w = \iota : (x, y) \to (x, -y)$ is called the hyperelliptic involution. It is well known that the hyperelliptic involution is central in $\mathrm{Aut}(S)$, the conformal automorphism group of S. We can get analogous results for C using Accola's theorem on strong branching [**1**]. In increasing order of generality we have:

(1) Hyperelliptic case: the involution $\iota : (x, y) \to (x, -y)$ is central in $\mathrm{Aut}(S)$.
(2) Prime order or super-elliptic case: $n = p$ is a prime and $\sigma > (p-1)^2$, then C is normal in $\mathrm{Aut}(S)$. If $f(x)$ is square-free then C is central (Accola [**1**]).
(3) Fully ramified or generalized super-elliptic case: $d_i = gcd(n, t_i) = 1$ for all i. If $\sigma > (n-1)^2$, then C is normal in $\mathrm{Aut}(S)$ (Kontogeorgis [**13**]).
(4) Weakly malnormal case: (definition to follow). If C is weakly malnormal in C and $\sigma > (n-1)^2$, then C is normal in $\mathrm{Aut}(S)$. (Broughton-Wootton [**4**])

The rest of this paper concerns our approach to the determination of exceptional automorphisms of groups in case 3 above. We focus on an overview of the problems and some calculations. The detailed proofs and complete results of the classification are beyond the scope of this paper and will appear in the forthcoming paper [**4**]. However, the examples of calculation of explicit automorphisms (Example 1.4) and a description of moduli spaces (Section 2.2) are not discussed in [**4**].

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1.2. Automorphism groups of cyclic n-gonal surfaces. There is a great deal of interest in the automorphism group $A = \mathrm{Aut}(S)$ of a cyclic n-gonal surface, especially the normal case. In the normal case A/C is an automorphism group of the sphere, one of five types of Platonic groups $\mathbb{Z}_k$, D_k, A_4, Σ_4, A_5. One "simply" solves an extension problem

$$C \quad \hookrightarrow \quad N \quad \twoheadrightarrow \quad K.$$

The automorphisms can be explicitly written down as birational transformations of $\mathbb{P}^2$. See Example 1.4 at the end of this section. Here are some sample works on the problem of determination of automorphism groups, in increasing generality on the properties of the cyclic action.

- The case $n = 2$ (hyperelliptic case) has been studied extensively: Brandt, Stichtenoff, Bujalance, Gamboa, Gromadzki, Shaska ([**5**], [**6**], [**15**]).
- The case where $n = 3$, (cyclic trigonal surfaces): Accola, Bujalance, Bujalance, Costa, Izquierdo, Martinez, Ying, ([**1**],[**7**], [**9**], [**19**]).
- The case where $n = p$, for p a prime: Bartolini-Costa-Izquierdo, Gonzalez-Diez, Wootton, ([**2**], [**12**], [**17**], [**18**]).
- General n where the cyclic n-gonal morphism $S \to S/C$ is fully ramified: Kontogeorgis, Broughton, Wootton ([**13**], [**4**]).
- General n with weak malnormality conditions: Broughton, Wootton, ([**4**]).

Next we set out some notation, definitions, and some facts. Let S be a cyclic n-gonal surface, then:

- σ denotes the genus of S.
- $C = \langle w \rangle$ is a cyclic group of automorphisms of S, of order n, such that S/C has genus zero.
- $A = Aut(S)$ is the group of automorphisms of S.
- $N = N_A(C)$ is the normalizer of C in A.
- The group $K = N/C$ acts on $S/C = \mathbb{P}^1$. If K is not trivial, then it must be one of the five Platonic types noted previously.

Finally, we define exceptional automorphisms.

Definition 1.2. Let S be a cyclic n-gonal surface with cyclic n-gonal group C and other notation be as immediately above. Then an automorphism in $h \in A - N$ is called an *exceptional* automorphism of S.

Example 1.3. Here are some low genus cyclic n-gonal surfaces. The two surfaces with exceptional automorphisms are the well known Klein's quartic and Bring's curve.

Table 1

genus	A	N	K	$\lvert A/N\rvert$	$\lvert C\rvert$	$(t_1,\ldots,t_s)$
3	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_2$	1	7	$(1,1,5)$
3	$PSL_2(7)$	$\mathbb{Z}_3 \ltimes \mathbb{Z}_7$	$\mathbb{Z}_3$	8	7	$(1,2,4)$
4	Σ_5	$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$\mathbb{Z}_4$	6	5	$(1,2,3,4)$
4	$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$\mathbb{Z}_4$	1	5	$(1,1,4,4)$
4	$\mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_3$	1	5	$(1,1,1,2)$

The two normalizers $\mathbb{Z}_4 \ltimes \mathbb{Z}_5$ are not isomorphic.

Ultimately, we want to determine the automorphism group of any cyclic n-gonal surface. We will restrict our attention to generalized super-elliptic surfaces. The normal case $A = N$ is computable using well-known extension methods for the exact sequence

$$C \hookrightarrow N \twoheadrightarrow K.$$

Assuming that S is a generalized super-elliptic surface, $N = A$ if $\sigma > (n-1)^2$. So for fixed n we want to determine the finite number of cases where $N < A$ with exceptional automorphisms. Noting that $d_i = 1$ in equation 1.2 then

$$\sigma = \frac{(n-1)(s-2)}{2}. \tag{1.3}$$

As $\sigma \leq (n-1)^2$ when there are exceptional automorphisms, then the number of branch points satisfies $s \leq 2n$.

The next example shows how the action of C on S and K on S/C can be used to give explicit formulas for the automorphisms of cyclic n-gonal surfaces. The question was posed by Emilio Bujalance and Peter Turbek at the Linköping conference in Emilio Bujalance's honour.

Example 1.4. Let us find equations and automorphisms for surfaces where N is non-abelian, $C = \mathbb{Z}_p$, $K = \mathbb{Z}_q$ where p, q are primes satisfying $q|(p-1)$, and there are some additional restrictions. Let $C = \langle w \rangle$. Since p and q are relatively prime then there is an $h \in N$ of order q such that h projects to a generator $\overline{h}$ in K. As conjugation by h is an automorphism of C then $hwh^{-1} = w^r$ where $r^q = 1 \bmod p$. Furthermore, $r^q - 1 = (1 + r + r^2 \cdots + \cdots r^{q-1})(1-r)$, so either $r = 1$ or $p|(1 + r + r^2 \cdots + \cdots r^{q-1})$. As N is non-abelian we conclude that $1 < r < q$ and $1 + r + r^2 \cdots + \cdots r^{q-1} = 0 \mod p$. Let us assume that the h-action on $\mathbb{P}^1$ is given by $h : x \to \zeta x$ where $\zeta = \exp(\frac{2\pi i}{q})$, and let us assume that the a_j constitute a single regular K-orbit, i.e., no a_j equals 0 or ∞. We may then assume that $a_j = \zeta^{j-1}$. Next, the rotation number of w at a point lying over $\overline{h}(a_j)$ is the rotation number of $hwh^{-1} = w^r$ at a point lying over a_j. Therefore $t_{j+1} = rt_j \bmod p$. As we shall see later we may assume that $t_1 = 1$, and hence $t_j = r^{j-1}$, $t = \sum t_j = 1 + r + r^2 \cdots + \cdots r^{q-1}$, divisible by p as required. The corresponding equation is

$$y^p = f(x) = \prod_{j=0}^{q-1} (x - \zeta^j)^{r^j}.$$

Now observe that

$$\begin{aligned} f(\zeta x) &= \prod_{j=0}^{q-1}(\zeta x-\zeta^j)^{r^j} = \left(\prod_{j=0}^{q-1}\zeta^{r^j}\right)(x-\zeta^{-1})\prod_{j=1}^{q-1}(x-\zeta^{j-1})^{r^j} \\ &= \zeta^t \frac{(x-\zeta^{q-1})^{r^q}}{(x-\zeta^{q-1})^{r^q-1}}\left(\prod_{j=1}^{q-1}(x-\zeta^{j-1})^{r^{j-1}}\right)^r \\ &= \left(\frac{\zeta^u}{(x-\zeta^{q-1})^v}\right)^p \left(\prod_{j=0}^{q-1}(x-\zeta^j)^{r^j}\right)^r \\ &= \left(\frac{\zeta^u}{(x-\zeta^{q-1})^v}\right)^p f^r(x) \end{aligned}$$

where $t = pu$ and $r^q - 1 = pv$. Now set

$$\begin{aligned} x' &= \zeta x, \\ y' &= \frac{\zeta^u y^r}{(x-\zeta^{q-1})^v} \end{aligned}$$

and we see that

$$(y')^p = f(x')$$

on the plane model of the curve. Let $\phi(x,y) = (x',y')$ be the corresponding birational transformation of $\mathbb{P}^2$. Then ϕ leaves the plane model invariant, projects to $\overline{h}$ on $\mathbb{P}^1$ and satisfies $\phi \circ w = w^r \circ \phi$, considering w as the map $(x,y) \to (x,\omega y)$ on $\mathbb{P}^2$.

REMARK 1.5. The general method for constructing the above defining equation and others is outlined in [**18**]. We repeated the construction here for completeness and clarity.

2. Super-elliptic surfaces

2.1. What are (generalized) super-elliptic surfaces? A super-elliptic surface is a cyclic n-gonal surface, where

- $n = p$, a prime,
- $f(x)$ is square free, (implies that C will be central in A),
- p need not divide the degree of $f(x)$.

We generalize this definition to non-prime cyclic groups and relax the square-free condition.

DEFINITION 2.1. Let S be a cyclic n-gonal surface, whose plane model satisfies the requirements given in given earlier. If $gcd(n,t_i) = 1$ for all t_i, or alternatively, if the degree of ramification of π over a_i equals n, then S is called a *generalized super-elliptic surface.*

REMARK 2.2. There is much interest – motivated by cryptography – in computing in the Jacobian of super-elliptic surfaces S for fields of prime characteristic. It is a generalization of elliptic cryptography. See the paper of Shaska [**15**] for references.

REMARK 2.3. Alternative conditions for being super elliptic are:

- $S^\nu \to S$ has a single point over each finite singularity $(a_i, 0)$.

- S has a cusp singularity $(a_i, 0)$, i.e., only one local branch.

Generalized super-elliptic surfaces satisfy the *weak malnormality* condition.

> *Weakly Malnormal Condition*: for all $g \in \mathrm{Aut}(S)$ either $gCg^{-1} = C$ or $gCg^{-1} \cap C = \{1\}$.

If the surface satisfies the weak malnormality condition and $\sigma > (n-1)^2$, then C is normal in $\mathrm{Aut}(S)$. Our eventual goal is to determine the automorphism group for weakly malnormal actions, the groundwork is laid out in the paper [**4**]. However, the general weakly malnormal case seems very difficult, so we are initially working on an easier but broadly interesting problem. We address this in the next section by showing that super-elliptic surfaces are fairly ubiquitous.

2.2. How many super-elliptic surfaces are there? Given our cyclic n-gonal equation

$$y^n = f(x) = \prod_{i=1}^{s} (x - a_i)^{t_i},$$

call $(a_1, a_2, \ldots, a_s)$ the branch points of S, $(t_1, t_2, \ldots, t_s)$ the multi-degree of S, and $(n_1, n_2, \ldots, n_s)$ – where $n_i = n/gcd(n, t_i)$ – the branching data or signature of the action of C on S. The following is easily shown

LEMMA 2.4. *Two surfaces with branch points* $(a_1, a_2, \ldots, a_s)$ *and* $(b_1, b_2, \ldots, b_s)$ *are conformally equivalent if there is an* $L \in PSL_2(\mathbb{C})$ *and a permutation* $\vartheta \in \Sigma_s$, *preserving multi-degree, so that*

$$b_i = L(a_{\vartheta i}).$$

for all i.

Let Σ_T denote the group of permutations preserving the multi-degree T. The lemma leads to the following statement.

> *Variety of surfaces of given multi-degree*: The variety
>
> $$\mathcal{MC}_{n,T} = (\mathbb{C}^s - diagonals)/(PSL_2(\mathbb{C}) \times \Sigma_T)$$
>
> of degree $s-3$ is "almost" a moduli space for the surfaces of multi-degree T.

Rather that make the notion "almost" precise enough to convert the statement into a proposition we note the following.

(1) Every cyclic n-gonal action with multi-degree $(t_1, t_2, \ldots, t_s)$ is accounted for in the quotient space.
(2) The action of $PSL_2(\mathbb{C})$ is only partial and exceptional automorphisms need to be taken into account. Additionally, as in Example 1.4, there may be an automorphism L mapping the branch point set $\{a_i\}$ to itself, for which the induced permutation of the degrees $\{t_i\}$ does not fix the multi-degree, even though the two surfaces are conformally equivalent.
(3) Each $\mathcal{MC}_{n,T}$ corresponds to a moduli space, of the same dimension, of Fuchsian groups determined by the signature $(n_1, n_2, \ldots, n_s)$. See the next section for the notation of signatures of Fuchsian groups.

In Table 1 we give the number of inequivalent multi-degrees for cyclic 35-gonal surfaces with 4 branch points. Only the first line corresponds to generalized super-elliptic surfaces.

(n_1,n_2,n_3,n_4)	# multidegrees	$lcm(n_1,n_2,n_3,n_4)$	genus
$(35,35,35,35)$	26	35	34
$(35,35,35,7)$	18	35	32
$(35,35,35,5)$	13	35	31
$(35,35,7,7)$	12	35	30
$(35,35,7,5)$	8	35	29
$(35,35,5,5)$	6	35	28
$(35,7,7,5)$	2	35	27
$(35,7,5,5)$	3	35	26
$(7,7,7,7)$	4	7	reducible
$(7,7,5,5)$	1	35	24
$(5,5,5,5)$	3	5	reducible

TABLE 1. Multi-degrees for cyclic 35-gonal surfaces with 4 branch points

3. Discovering exceptional automorphisms

Determination of the full automorphism group of a cyclic n-gonal surface requires more work than lifting the action of K on S/C and requires an entirely different method than that suggested in Example 1.4. We first discuss a mechanism that encodes the information in the triple $C \trianglelefteq N < A$. The proofs and more details of the results of this section are given in [**4**].

3.1. Covering actions by Fuchsian groups. A (co-compact) Fuchsian group Γ, a discrete group acting on the hyperbolic plane $\mathbb{H}$, has a presentation by hyperbolic and elliptic generators and relations:

$$\text{generators}: \{\alpha_i, \beta_i, \gamma_j, 1 \le i \le \sigma, 1 \le j \le s\}$$

$$\text{relations}: \prod_{i=1}^{\sigma}[\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_s^{m_s} = 1$$

The signature of Γ is

$$\mathcal{S}(\Gamma) = (\sigma : m_1, \ldots, m_s)$$

$(= (m_1, \ldots, m_s)$, when the genus is zero). Here are important invariants of a Fuchsian group, they are all derived from the signature.

- *genus of* Γ: $\sigma(\Gamma) = \sigma$ is the genus of $\mathbb{H}/\Gamma$.
- The area of a fundamental region of Γ is given by $A(\Gamma) = 2\pi\mu(\Gamma)$ where,
$$\mu(\Gamma) = 2(\sigma - 1) + \sum_{j=1}^{s}\left(1 - \frac{1}{m_j}\right).$$
- The *Teichmüller dimension* $d(\Gamma)$ of Γ, the dimension of the Teichmüller space of Fuchsian groups with signature, $\mathcal{S}(\Gamma)$ given by
$$d(\Gamma) = 3(\sigma - 1) + s.$$

For S a cyclic n-gonal surface, we have a covering diagram

(3.1)
$$\begin{array}{ccccc} \Gamma_C & \hookrightarrow & \Gamma_N & \hookrightarrow & \Gamma_A \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ C & \hookrightarrow & N & \hookrightarrow & A \end{array}$$

for an exact sequence

$$\Pi \hookrightarrow \Gamma_A \overset{\eta}{\twoheadrightarrow} A \tag{3.2}$$

such that Π is torsion free and $S = \mathbb{H}/\Pi$. By considering covering triples $\Gamma_C \trianglelefteq \Gamma_N < \Gamma_A$, we may classify $C \trianglelefteq N < A$ using computational group theory methods as follows.

- Determine the Fuchsian group signature triples allowed.
- From the signatures, determine the possible epimorphisms $\eta : \Gamma_C \to C$ and the K-action on $\mathbb{P}^1$. If the K-action is compatible with the epimorphism η then $\Gamma_C \trianglelefteq \Gamma_N$.
- Determine monodromy of Fuchsian group pairs $\Gamma_C \trianglelefteq \Gamma_N$, $\Gamma_N < \Gamma_A$ (definition next section)
- Determine the "word maps" of Fuchsian group pairs $\Gamma_C \trianglelefteq \Gamma_N$, $\Gamma_N < \Gamma_A$ (definition next section)
- Using the monodromy and word maps, the monodromy of the pairs $C \trianglelefteq N$ and $N < A$, may be fused together to produce A.

3.2. Restrictions on structure and signatures. Here we indicate how the weakly malnormal and super-elliptic conditions limit the structure the triple $C < N < A$ and the signatures of the possible triples.

- The weakly malnormal and super-elliptic conditions limit the structure of the signatures of the possible triples.
- There are finitely many cases of parametric families and finitely many exceptional signature pairs to consider.
- The two different conditions (weakly malnormal and super-elliptic) require different computational methods.

The following theorem describes parametric families of surfaces [**4**].

THEOREM 3.1. *If S is generalized super-elliptic (or even if C has a weakly malnormal action), and N is not self-normalizing in A then N contains a copy of $\mathbb{Z}_n \times \mathbb{Z}_n$ and there are three possibilities given in Table 2.*

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$m = \lvert\Gamma_A/\Gamma_N\rvert$	$\lvert C\rvert$	genus	Group
$(2,3,2n)$	$(2,n,2n)$	3	$n \geq 5$	$\frac{(n-1)(n-2)}{2}$	$\Sigma_3 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,2,2,n)$	$(2,2,n,n)$	2	$n \geq 3$	$(n-1)^2$	$V_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,4,2n)$	$(2,2n,2n)$	2	$n \geq 3$	$(n-1)^2$	$D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

TABLE 2. Generalized super-elliptic surfaces for non-normalizing N

THEOREM 3.2. *If S is weakly malnormal then Γ_N has at most 3 more periods than Γ_A. If Γ_A and Γ_N have the same number of canonical generators, then they appear in Singerman's list. The signatures for Γ_A and Γ_N appear as a pair in Table 3. In the table (a_1, a_2, a_3) or (k, k) is the signature of $K = \Gamma_N/\Gamma_C$, as spherical crystallographic group. The signature for Γ_C is automatically determined from Γ_N. The type is of the form cA or cB, where c is the Teichmüller codimension (difference in the number of generators) and A denotes a non-cyclic K and B denotes a cyclic K.*

Type	Signature of Γ_N	Signature of Γ_A
0A	$(0; a_1m_1, a_2m_2, a_3m_3, , n_1, \ldots, n_r)$	$(0; b_1, b_2, b_3, n_1, \ldots, n_r)$
0B	$(0; km_1, km_2, , n_1, \ldots, n_r)$	$(0; b_1, b_2, n_1, \ldots, n_r)$
1A	$(0; a_1m_1, a_2m_2, a_3m_3, , n_1, \ldots, n_r)$	$(0; b_1, b_2, n_1, \ldots, n_r)$
1B	$(0; km_1, km_2, , n_1, \ldots, n_r)$	$(0; b_1, n_1, \ldots, n_r)$
2A	$(0; a_1m_1, a_2m_2, a_3m_3, , n_1, \ldots, n_r)$	$(0; b_1, n_1, \ldots, n_r)$
2B	$(0; km_1, km_2, , n_1, \ldots, n_r)$	$(0; n_1, \ldots, n_r)$
3A	$(0; a_1m_1, a_2m_2, a_3m_3, , n_1, \ldots, n_r)$	$(0; n_1, \ldots, n_r)$

TABLE 3. Signatures for weakly malnormal actions

THEOREM 3.3. *If S is a generalized super-elliptic surface, n is composite, and N is self-normalizing in A then the signatures of Γ_N, and Γ_A are as in the Tables 4 and 5. The examples in Table 4 occur in Singerman's list, those in Table 5 have positive codimension. The type notation cA or cB is as described in Theorem 3.2 and the quantities m_i must be 1 or n.*

REMARK 3.4. Additional constraints on the signatures in Table 5 have not been determined at the time of writing this paper.

Type	$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$m = \|\Gamma_A/\Gamma_N\|$	K	n
0A	$(2, 3, 8)$	$(3, 8, 8)$	10	D_3	4
0A	$(2, 3, 8)$	$(2, 8, 8)$	6	D_8	4
0A	$(2, 3, 4q)$	$(q, 4q, 4q)$	6	D_q	$2q$
0A	$(2, 4, 2q)$	$(q, 2q, 2q)$	4	D_q	q
0A	$(2, 3, 12)$	$(3, 4, 12)$	4	Σ_4	6
0A	$(2, 3, 2q)$	$(2, q, 2q)$	3	D_q	q
0B	$(2, 3, 8)$	$(4, 8, 8)$	12	$\mathbb{Z}_2$	4
0B	$(2, 3, 9)$	$(9, 9, 9)$	12	$\mathbb{Z}_9$	9
0B	$(2, 3, 9)$	$(3, 3, 9)$	4	$\mathbb{Z}_3$	9

TABLE 4. Co-dimension 0 – from Singerman's list

Type	$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$m = \|\Gamma_A/\Gamma_N\|$
1A	(a_1n, a_2n, n)	(a_1n, a_2n, a_3n_3, n)	$m \leq 6$
1A	(a_1n, b_2, n)	$(a_1n, a_2n_2, a_3n_3, n)$	$m \leq 8$
1B	(b_1, n, n)	(k, k, n, n)	$m \leq 5$
2A	(b_1, b_2, n)	(a_1, a_2, a_3, n)	$m \leq 20$
2A	(b_1, b_2, n, n)	$(a_1m_1, a_2m_2, a_3m_3, n, n)$	$m = 3$
2A	(a_1n, n, n)	(a_1n, a_2n, a_3, n, n)	$m \leq 7$
2A	(a_1n, n, n)	(a_1n, a_2, a_3, n, n)	$m \leq 5$
2A	(b_1, n, n)	(a_1, a_2, a_3, n, n)	$m \leq 10$
2B	$(4, 4, 4)$	$(2, 2, 4, 4, 4)$	$m = 5$
3A	(n, n, n)	(a_1, a_2, a_3, n, n, n)	$m < 3 + \frac{6}{n-3}$

TABLE 5. Co-dimension > 0

EXAMPLE 3.5. Table 6 shows some examples of known generalized super-elliptic surfaces. The examples show prime cases, special cases, and families.

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$m = \lvert\Gamma_A/\Gamma_N\rvert$	$\lvert C\rvert$	genus	Group
$(2,3,4n)$	$(2,2,3,n)$	4	$n=2$	2	$GL(2,3)$
$(2,3,7)$	$(3,3,7)$	8	7	3	$PSL_2(7)$
$(2,3,3n)$	$(3,n,3n)$	4	$n=3$	3	$\mathbb{Z}_4.A_4$
$(2,4,5)$	$(4,4,5)$	6	5	4	Σ_5
$(2,3,2n)$	$(2,n,2n)$	3	$n\geq 5$	$\frac{(n-1)(n-2)}{2}$	$\Sigma_3 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,2,2,n)$	$(2,2,n,n)$	2	$n\geq 3$	$(n-1)^2$	$V_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,4,2n)$	$(2,2n,2n)$	2	$n\geq 3$	$(n-1)^2$	$D_4 \ltimes (\mathbb{Z}_n \ltimes \mathbb{Z}_n)$

TABLE 6. Known families of generalized super-elliptic surfaces

The group $G = \mathbb{Z}_4.A_4$ in the third case above, is a central extension of A_4 by $\mathbb{Z}_4$. In the Magma database $G = \mathrm{SmallGroup}(48,33)$ is group number 33 in the groups of order 48.

4. Classification overview

4.1. Fuchsian group pairs. Suppose that $\Gamma < \Delta$ is a Fuchsian group pair. We also suppose that Γ has genus σ and s elliptic generators and that Δ has genus τ and t elliptic generators. For notational convenience, we denote the canonical generating sets of Γ and Δ, respectively, by:

$$\mathcal{G}_1 = \{\theta_1, \ldots, \theta_{2\sigma+s}\}$$

and

$$\mathcal{G}_2 = \{\zeta_1, \ldots, \zeta_{2\tau+t}\},$$

We call the quantity

$$c(\Gamma, \Delta) = d(\Gamma) - d(\Delta)$$

the *Teichmüller codimension* of the pair $\Gamma < \Delta$. In any calculation we will always assume that $\sigma = \tau = 0$ so that $c(\Gamma, \Delta) = s - t$, the difference in the number of generators.

We next discuss two ways of describing the inclusion $\Gamma < \Delta$ namely monodromy groups and word maps.

4.1.1. *Monodromy groups.* The pair $\Gamma < \Delta$ determines a permutation or monodromy representation of Δ on the cosets of Γ

$$\rho : \Delta \to \Sigma_m$$

where m is the index of Γ in Δ. Write

$$\mathcal{P} = (\pi_1, \pi_2, \ldots, \pi_{2\tau+t})$$

for $\pi_i = \rho(\zeta_i) \in \Sigma_m$, to construct the *monodromy vector* of the pair. The cycle types and other properties of $\mathcal{P}$ are determined by signatures $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\Delta)$ and the relations on the generators.

Definition 4.1. Let notation be as above. Then the *monodromy group* of the pair is defined by

$$M(\Delta,\Gamma)=\rho(\Delta)=\langle\pi_1,\pi_2,\ldots,\pi_{2\tau+t}\rangle$$

It is well defined up to conjugacy in Σ_m. We extend the definition of monodromy of a group pair to the sequence $C<N<A$. Then we have these relations because of the diagram 3.1

$$\begin{aligned}M(\Gamma_A,\Gamma_N)&=M(A,N)\\M(\Gamma_N,\Gamma_C)&=M(N,C)=M(N/C,\langle 1\rangle)\simeq K\\M(\Gamma_A,\Gamma_C)&=M(A,C)\simeq A.\end{aligned}$$

The last equation holds because of the super-elliptic or weak malnormality condition both of which implies that $\bigcap_{g\in A} gCg^{-1}=\langle 1\rangle$ since A strictly contains N.

4.1.2. *Word Maps.* The inclusion $\Gamma<\Delta$ may also be described by a word map which we define now.

Definition 4.2. Let notation be as above. Then the *word map* of the inclusion $\Gamma\hookrightarrow\Delta$ (with respect to the given generating sets) is a set of words $\{w_1,\ldots,w_{2\sigma+s}\}$ in the generators in $\mathcal{G}_2$ such that

$$\theta_i=w_i(\zeta_1\ldots,\zeta_{2\tau+t}),i=1,\ldots,2\tau+t.$$

We note that given a word map for the inclusion $\Gamma\hookrightarrow\Delta$ a monodromy vector $\mathcal{P}$ is easily calculated using the Todd-Coxeter algorithm. Given a monodromy vector $\mathcal{P}$ of a genus zero pair $\Gamma<\Delta$ (both groups), then the word map of the pair may be calculated, in a standard way, using the Reidemeister-Schreier method. See for instance [**10**], [**8**] and [**11**].

Example 4.3. Suppose we have the following signatures

$$\mathcal{S}_1=(2,2,2,5),\mathcal{S}_2=(2,4,5)$$

We want a pair $\Gamma<\Delta$ with

$$\mathcal{S}(\Gamma)=\mathcal{S}_1,\mathcal{S}(\Delta)=\mathcal{S}_2$$

First we find a compatible monodromy vector in Σ_6

$$\pi_1=(1,3)(4,6),\pi_2=(1,2)(3,5,4,6),\pi_3=(1,2,3,4,5),$$

note that $M(\Delta,\Gamma)=A_6$. Define $\rho\colon\Delta\to\Sigma_6$ by $\rho\colon\zeta_i\to\pi_i, i=1\ldots 3$ and set Γ to be the stabilizer of a point for the permutation action of Δ on $\{1,\ldots,6\}$. From the algorithm, a generating set for Γ is:

$$\begin{aligned}\theta_1&=(\zeta_1\zeta_2)\zeta_1(\zeta_1\zeta_2)^{-1}\\\theta_2&=\zeta_2\zeta_1\zeta_2^{-1}\\\theta_3&=\zeta_2^2\\\theta_4&=(\zeta_2^{-1}\zeta_1^{-1}\zeta_2^{-1}\zeta_1\zeta_3\zeta_1)\zeta_3(\zeta_2^{-1}\zeta_1^{-1}\zeta_2^{-1}\zeta_1\zeta_3\zeta_1)^{-1}\end{aligned}$$

4.2. Steps of the classification. The discussion following equation 3.2 may be expanded to classify all super-elliptic surfaces with exceptional automorphisms. Full results will appear in [**4**].

(1) Using Magma or GAP determine all signature pairs $\mathcal{S}(\Gamma_N)$ and $\mathcal{S}(\Gamma_A)$ for codimension 0,1,2,3 treating special pairs and families of pairs separately. In the super-elliptic case the search is considerably cut down because of the restriction on signatures.
(2) The group K and the signature $\mathcal{S}(\Gamma_C)$ is automatically determined. In the super-elliptic case all the periods equal n and only the number of periods needs to be determined.
(3) The action of the group K on $\mathbb{P}^1$ and the possible epimorphisms $\eta : \Gamma_C \to C$ can be determined. The K-action and the epimorphism η must be compatible. If no such compatible action exists then there is no $\Gamma_C \trianglelefteq \Gamma_N$.
(4) For each candidate signature pair, compute all the compatible monodromy vectors up to conjugacy. Use the classification of primitive permutation groups (Magma or GAP).
(5) From monodromy vectors of $\Gamma_N < \Gamma_A$ and $\Gamma_C \trianglelefteq \Gamma_N$ compute the word maps of $\Gamma_C \hookrightarrow \Gamma_N$ and $\Gamma_N \hookrightarrow \Gamma_A$
(6) Compute the word map of $\Gamma_C \hookrightarrow \Gamma_A$ by substitution.
(7) Compute the monodromy group $M(\Gamma_A, \Gamma_C)$ using the Todd-Coxeter algorithm.
(8) If the stabilizer of a point in $M(\Gamma_A, \Gamma_C) \simeq A$ is not cyclic then reject this case.

In the weakly malnormal case there are 202 special pairs and 597 families of pairs $\Gamma_N < \Gamma_A$ that could potentially lead to cyclic n-gonal surfaces. Obviously this cannot be done by hand. In the super-elliptic case computer calculation is still needed but the number of cases is dramatically reduced.

Let us finish with some simple calculations that illustrate steps 1-4. Actual calculation with words maps are cumbersome and beyond the space limitations of this paper. Full details of all calculations will appear in [**4**].

Example 4.4. Let us consider the last line of Table 4 with $\mathcal{S}(\Gamma_A) = (2,3,9)$, $\mathcal{S}(\Gamma_N) = (3,3,9)$, $m = |\Gamma_A/\Gamma_N| = 4$, $K = \mathbb{Z}_3$ and $n = |C| = 9$. We see from the table that K has two fixed points of order 3 on S/C and that the branch points of the C-action from a regular K-orbit. The only possibilities for the multi-degree (t_1, t_2, t_3) are $(1,1,7)$, $(1,4,4)$, Now the multi-degree must satisfy $t_{i+1} = rt_i$ for some r satisfying $r^3 = 1 \bmod 9$. This is not possible and we stop here. We do note that the pair $\Gamma_N < \Gamma_A$ does exist. For, we see that the monodromy vector $\mathcal{P} = (\pi_1, \pi_2, \pi_3)$ must have cycle types 2^2, $1 \cdot 3$, and $1 \cdot 3$ respectively. We may choose $\pi_1 = (1,3)(2,4)$, $\pi_2 = (1,2,3), \pi_3 = (2,3,4)$.

Example 4.5. Let us consider the third last line of Table 4 with $\mathcal{S}(\Gamma_A) = (2,3,8)$, $\mathcal{S}(\Gamma_N) = (4,8,8)$, $m = |\Gamma_A/\Gamma_N| = 12$, $K = \mathbb{Z}_2$ and $n = |C| = 4$. We see from the table that K has two fixed points of order 2 on S/C and so that the four branch points of order 4 must lie on the K-fixed points and on one regular K-orbit. The possibilities for the multi-degree (t_1, t_2, t_3, t_4) are $(1,1,1,1)$ and $(1,1,3,3)$ permutations if t_1 and t_2 correspond to fixed points. Thus one can proceed to find permutations. The cycle types must be 2^6, $3^4 \cdot 3$, and $1^2 \cdot 2 \cdot 8$. If we are looking for pairs $\Gamma_N < \Gamma_A$ with primitive $M(\Gamma_A, \Gamma_N)$ then there is a monodromy vector $\mathcal{P} = (\pi_1, \pi_2, \pi_3)$ of the prescribed cycle type in the Mathieu group M_{12} of order 95040. Thus there is a pair $\Gamma_N < \Gamma_A$ with $M(\Gamma_A, \Gamma_N) \backsimeq M_{12}$. However, as $M(\Gamma_A, \Gamma_N)$ is an image of A then, $\frac{|A|}{|M(\Gamma_A,\Gamma_N)|}$ is an integer. But,

$$\frac{|A|}{|M(\Gamma_A, \Gamma_N)|} = \frac{\frac{|A|}{|N|} \times \frac{|N|}{|C|} \times |C|}{|M(\Gamma_A, \Gamma_N)|} = \frac{m \times |K| \times n}{|M_{12}|} = \frac{12 \times 2 \times 4}{95040} = \frac{1}{990}.$$

The other primitive groups can be rejected for other reasons, and so one must look among the 301 transitive groups of degree 12.

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